NS-364B

Instituut voor Theoretische Fysica, Universiteit Utrecht

EXAM CLASSICAL FIELD THEORY

April 20, 2010

- The duration of the test is 3 hours.
- Only the lecture notes by Gleb Arutyunov *"Classical Field Theory"* may be consulted during the test.
- Usage of a calculator and a dictionary is allowed
- Use different sheets for each exercise.
- Write your name and initials on every sheet handed in.
- Divide your available time wisely over the exercises.

Problem 1

Consider the Lagrangian for the particle in d-dimensions with coordinates $q_i(t)$, $i = 1, \dots d$, moving in some potential

$$L = \sum_{i=1}^{d} \left(\frac{m}{2} \dot{q}_i^2 - w^2 q_i^2 - \lambda (q_i^2)^2 \right).$$

1. Argue that the Lagrangian is invariant under d-dimensional rotation of coordinates $q_i \to q_i' = O_{ij} q_j$, where O is an arbitrary orthogonal constant matrix. Derive the Noether currents corresponding to these symmetry transformations. How many independent Noether currents do you find?

2. Find the corresponding Hamiltonian.

Problem 2

Consider a scalar field $u(x,t)$ satisfying the Korteweg-de Vries (KdV) equation

$$u_t + 6uu_x + u_{xxx} = 0$$

and the periodicity condition $u(x + 2\pi) = u(x)$. Here $u_t \equiv \frac{\partial u}{\partial t}$, $u_x \equiv \frac{\partial u}{\partial x}$ and $u_{xxx} \equiv \frac{\partial^3 u}{\partial x^3}$.

1. Find a value of the constant α for which the following functional

$$H[u] = \int_0^{2\pi} \mathrm{d}x (u^3 + \alpha u_x^2)$$

is the Hamiltonian[1] giving rise to the KdV equation, *i.e.* $u_t = \{H, u\}$, with respect to the following Poisson bracket (the Gardner bracket)

$$\{F, G\} = \int_0^{2\pi} \mathrm{d}x \frac{\delta F}{\delta u(x)} \frac{\partial}{\partial x} \left(\frac{\delta G}{\delta u(x)} \right) ,$$

where F, G are two arbitrary functionals of u.

2. Show that the Gardner bracket is skew-symmetric and satisfies the Jacobi identity.

Problem 3

Consider the action for electromagnetic potential A_μ, $\mu = 0, 1, 2, 3$

$$S = -\frac{1}{4} \int \mathrm{d}^4 x \, F_{\mu\nu} F^{\mu\nu} ,$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Which symmetries of this action do you know? Use this action to obtain the equations of motion for A_μ. Find the Noether currents corresponding to the Lorentz symmetry transformations which have the following infinitezimal form

$$x^\mu \to x^\mu + \delta x^\mu , \qquad \delta x^\mu = \Lambda^{\mu\nu} x_\nu , \qquad \Lambda^{\mu\nu} = -\Lambda^{\nu\mu} .$$

The indices are lowered and raised with the help of the Minkowski metric.

[1] Do not be surprised that this Hamiltonian has unusual form, *i.e.* it is not written in terms of the canonical momentum. As you can see, the Poisson bracket is also non-canonical.

Problem 4

An averaged (in time) potential φ for the neutral hydrogen atom is described by the following formula

$$\varphi = e\frac{e^{-\alpha r}}{r}\left(1 + \frac{\alpha r}{2}\right),$$

where e is the charge of electron and $\alpha = \frac{2}{a_0}$, and $a_0 = \frac{e^2}{mc^2}$ is the so-called classical radius of electron (m is the mass of electron and c is the speed of light). Find the charge density distribution which leads to such a potential and explain its physical meaning.

Problem 5

Which transformations of phase space coordinates are called canonical?