NS - 364B

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EXAM CLASSICAL FIELD THEORY

June 29, 2010

- The duration of the test is 3 hours.
- Only the lecture notes by Gleb Arutyunov *"Classical Field Theory"* may be consulted during the test.
- Usage of a calculator and a dictionary is allowed
- Use different sheets for each exercise.
- Write your name and initials on every sheet handed in.
- Divide your available time wisely over the exercises.

Problem 1

Consider the following Lagrangian density for a massive vector field

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{m^2}{2}A_\mu A^\mu .$$

Show that every component A_μ satisfies the Klein-Gordon equation

$$(\partial_\nu\partial^\nu + m^2)A_\mu = 0 .$$

Problem 2

The Hamiltonian for a real two-dimensional scalar field $\phi(x,t)$ is given by

$$H = \int \mathrm{d}x \left[\frac{1}{2}p^2 - \frac{1}{2}\partial_x\phi\,\partial_x\phi - \frac{m^2}{\beta^2}(1-\cos\beta\phi)\right],$$

where m is the mass and β is a parameter (the coupling constant). By using the Noether theorem Find the momentum P corresponding to the space translations and the generator K of a Lorentz rotation:

$$x^\mu \to x^{\mu'} = x^\mu + a^\mu$$
$$x^\mu \to x^{\mu'} = \Lambda^\mu{}_\nu x^\nu , \quad \Lambda^{\mu\nu} = -\Lambda^{\nu\mu} .$$

Check that the found quantities do not depend on time.

Problem 3

Consider the action for electromagnetic field A_μ:

$$S = -\frac{1}{4}\int \mathrm{d}^4x \, F_{\mu\nu}F^{\mu\nu} .$$

1. Derive the corresponding Euler-Lagrange equations;
2. Enumerate global and local symmetries of this action;
3. Derive the corresponding stress-energy tensor.

Problem 4

Concerning the Lorentz group, answer the following questions:

1. How many connected components has the Lorentz group?
2. How these components are related to each other?
3. Which of these components is a subgroup of the Lorentz group?

Problem 5

Consider the following vector and scalar potentials

$$\vec{A}(x,t) = \vec{A}_0 e^{i(\vec{k}\cdot\vec{x}-\omega t)} , \qquad \varphi(x,t) = 0 ,$$

where $\vec{A}_0$ and $\vec{k}$ are constant three-dimensional vectors, and ω is a constant frequency.

1. Derive the electric and magnetic fields corresponding to these potentials
2. Determine the conditions imposed on $\vec{A}_0$, $\vec{k}$ and ω by Maxwell's equations assuming that the absence of charge and current densities, *i.e.* $\rho = 0$ and $\vec{j} = 0$.

Problem 6 (Bonus)

Consider an interaction scalar field $\phi(x)$ on the $3+1$-dimensional Minkowski space-time described by the following action

$$S = \int \mathrm{d}^4x \Big(\frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi - \lambda \phi^4 \Big) ,$$

where λ is a constant (called the "coupling constant").

1. Show that the action is invariant under infinitezimal transformations

$$\phi \to \phi + \delta\phi \,, \qquad \delta\phi = \epsilon (x^\mu \partial_\mu \phi + \phi) \,,$$

   up to a total derivative term $\delta S = \epsilon \int \mathrm{d}^4 x \; \partial_\mu F^\mu$, where ϵ is a constant small (infinitezimal) parameter. Find the vector F^μ explicitly.

2. Construct the corresponding Noether current J^μ by using the general expression from the lecture notes. You can check, however, that this current is not conserved, *i.e.* $\partial_\mu J^\mu \neq 0$. The reason for this non-conservation is that the action S is not *exactly* invariant under infinitezimal transformations of ϕ, but it is invariant up to a total derivative term.

3. Show that an improved current

$$J^\mu_{\text{improved}} = J^\mu + F^\mu$$

   is conserved due to equations of motion.