Classical field theory 2012 (NS-364B) – Final exam

Tue June 26 2012, 15:00-18:00 MG kantine. Assistent: Laurent Dufour. In total 60 points = 60%
You have *three* hours to solve this exam. The exam is closed books. *Good luck!*

Problem 1. Theoretical questions. *(12 points)*

(A) Electromagnetism can be defined in terms of the vector potential A^μ, which has four components. Because of the gauge symmetry, not all components of A^μ are physical (observable). How many components of A^μ are physical? How many of them are dynamical (propagating)?

(B) Feynman propagator is not causal. You have seen it explicitly by constructing it for a massless scalar field, $\imath\Delta_F(x;x') = -[1/(4\pi^2\Delta x^2_{++})]$, where $\Delta x^2_{++} = (|t-t'|-\imath\epsilon)^2 - \|\vec{x}-\vec{x}'\|^2$ is a complexified invariant distance function. Identify the causal part of the Feynman propagator, and the part that is not causal. Explain what each part means. *Hint:* Make use of the Dirac (or Sokhotski-Plemelj) identity.

(C) Define Coulomb gauge in electromagentism.

(D) Write down the 'constraint' Maxwell equations for the electric and magnetic fields. *Hint:* These are the equations that correspond to $\partial_\mu \tilde{F}^{\mu\nu} = 0$, where $\tilde{F}^{\mu\nu}$ is a dual field strength tensor.

Problem 2. A particle in constant magnetic field. *(6 points)*
By solving the suitable relativistic equation of motion, calculate the velocity $\vec{v}(t)$ of a relativistic particle in a constant magnetic field $\vec{B}$. This particle is moving on a circle with a certain angular frequency. Calculate the radius of the circle and the frequency ω. What is the common name of this frequency?
Hint: Take for simplicity that $\vec{B}$ points in the $\hat{z}$-direction. Recall that the action for a relativistic particle in an external electromagentic field is,

$$S[A^\mu, x^\mu] = -mc\int ds - \frac{e}{c}\int A_\mu dx^\mu\,. \qquad (1)$$

Problem 3. Maxwell's equations. *(4 points)*
Prove that $\partial_\mu \tilde{F}^{\mu\nu} = 0$, where $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ is the dual field strength tensor.

Problem 4. Lorentz transformations *(8 points)*
A point static charge q produces the following electromagnetic potential,

$$A^0 \equiv \varphi(\vec{x}) = \frac{q}{r}\,, \qquad \vec{A}(\vec{x}) = 0\,. \qquad (2)$$

(A) *(4 points)* By making use of an appropriate Lorentz boost, construct $(\varphi', \vec{A}')$ produced by a charge moving with a constant velocity.

Hint: Take for simplicity that the charge is moving in the $\hat{x}$-direction.

(B) *(4 points)* Derive the corresponding electric and magnetic fields.

Hint: If you have forgotten explicit formulae for $\vec{E}$ and $\vec{B}$ in terms of A^μ, recall the $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ contains electric and magnetic fields in its F_{0i} and F_{ij} components, respectively. As usually, i, j, k are spatial and μ, ν are space-time indices.

Problem 5. Liénard-Wiechert potentials. *(13 points)*

(A) *(5 points)*

Derive the following expressions for Liénard-Wiechert potentials of a charged particle moving on an arbitrary trajectory $\vec{r} = \vec{r}(t)$,

$$\varphi(\vec{x}, t) = \frac{e}{R - \vec{v} \cdot \vec{R}/c}\,, \qquad \vec{A}(\vec{x}, t) = \frac{e\vec{v}/c}{R - \vec{v} \cdot \vec{R}/c}\,, \tag{3}$$

where e is particle's charge, $\vec{R}(\vec{x}, t') = \vec{x} - \vec{r}(t')$ is the distance from the observer to the particle, $\vec{r}(t')$ is particle's trajectory, $\vec{v}(t) = d\vec{r}(t')/dt'$ is its velocity and c denotes the speed of light. Recall that in Eq. (3) $t' = t'(\vec{x}, t)$ is the retarded time implicitly given by $t' = t - R(\vec{x}, t')/c$.

Hint: You may use the integral formulae for φ and $\vec{A}$ in Lorenz gauge, in terms of ρ and $\vec{j}$,

$$\varphi(\vec{x}, t) = \int d^3x' \frac{\rho(\vec{x}', t')}{\|\vec{x} - \vec{x}'\|}\,, \qquad \vec{A}(\vec{x}, t) = \frac{1}{c}\int d^3x' \frac{\vec{j}(\vec{x}', t')}{\|\vec{x} - \vec{x}'\|}\,, \tag{4}$$

where $t' = t - \|\vec{x} - \vec{x}'\|/c$ is the retarded time. Recall further that the current density of a moving charge e can be written as,

$$\rho(\vec{x}, t) = e\delta^3(\vec{x} - \vec{r}(t))\,, \qquad \vec{j}(\vec{x}, t) = e\vec{v}\delta^3(\vec{x} - \vec{r}(t))\,, \tag{5}$$

where $\vec{v}(t) = d\vec{r}(t)/dt$ and $\vec{r} = \vec{r}(t)$ is particle's trajectory.

(B) *(3 points)* The corresponding magnetic and electric fields are given by,

$$\vec{E} = \frac{e}{R^2}\frac{\left(1 - \frac{v^2}{c^2}\right)\left(\frac{\vec{R}}{R} - \frac{\vec{v}}{c}\right)}{\left(1 - \frac{\vec{v}\cdot\vec{R}}{cR}\right)^3} + \frac{e}{c^2 R}\frac{\frac{\vec{R}}{R} \times \left[\left(\frac{\vec{R}}{R} - \frac{\vec{v}}{c}\right) \times \vec{a}\right]}{\left(1 - \frac{\vec{v}\cdot\vec{R}}{cR}\right)^3}\,, \qquad \vec{B} = \frac{\vec{R}}{R} \times \vec{E}\,, \tag{6}$$

where $\vec{a}(t') = d\vec{v}/dt'$ is particle's acceleration at a time $t' = t'(\vec{x}, t)$. Identify which parts of the expressions in (6) dominate in the near zone and which parts in the radiation (or wave) zone. Compare these expressions to determine the radius of the sphere which divides the radiation from the near zone. Give an estimate for the radius of that sphere!

(C) *(4 points)* Calculate the Poynting vector in the near zone,

$$\vec{S}_{\text{near}} = \frac{c}{4\pi}\vec{E}_{\text{near}} \times \vec{B}_{\text{near}}\,. \tag{7}$$

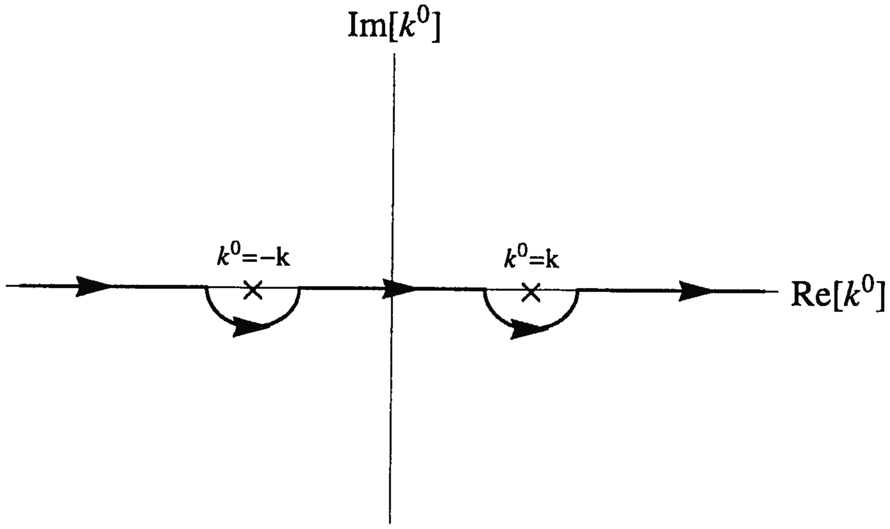

Figure 1: The integration contour in complex k^0-plane for the advanced propagator in problem 6.

Problem 6. The advanced Green function. *(17 points)*
Consider an action of a massless real scalar field ϕ,

$$S[\phi] = \int d^4x \left(\frac{1}{2}(\partial_\mu\phi)(\partial_\nu\phi)\eta^{\mu\nu}\right), \qquad \eta^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)\,. \tag{8}$$

The advanced Green function satisfies the following equation,

$$-\partial^2 \imath\Delta_a(x;x') = -\partial'^2 \imath\Delta_a(x;x') = \imath\hbar\delta^4(x-x')\,, \qquad \partial^2 = \eta^{\mu\nu}\partial_\mu\partial_\nu = \frac{1}{c^2}\partial_t^2 - \nabla^2\,. \tag{9}$$

(A) *(8 points)* By making use of the contour in complex k^0-plane shown in figure 1, construct the advanced Green function for the free scalar field whose action is given by (8).

(B) *(3 points)* The positive frequency Wightman function $\imath\Delta^+(x;x') \equiv \imath\Delta^{-+}(x;x')$ is given by $\imath\Delta^{-+}(x;x') = -\frac{\hbar}{4\pi^2}\frac{1}{\Delta x_{-+}^2}$, where $\Delta x_{-+}^2 = (t-t'-\imath\epsilon)^2 - \|\vec{x}-\vec{x}'\|^2$. By making use of the relation,

$$\imath\Delta_a(x;x') = \imath\Delta_F(x;x') - \imath\Delta^{-+}(x;x')\,, \tag{10}$$

construct the Feynman propagator and show that it equals to, $\imath\Delta_F(x;x') = -\frac{\hbar}{4\pi^2}\frac{1}{\Delta x_{++}^2}$, where $\Delta x_{++}^2 = (|t-t'|-\imath\epsilon)^2 - \|\vec{x}-\vec{x}'\|^2$.

(D) *(3 points)*

Combine in the right way the contours of integration in parts A and B to obtain the contour of integration in the complex k^0-plane for the Feynman propagator (10).

(E) *(3 points)*

Discuss the causality structure of the advanced Green function obtained in part A. *Hint:* Draw a light cone, and indicate where the advanced Green function does and where it does not vanish. Comment on what that implies for causality in relativistic field theory.