Institute for Theoretical Physics
Utrecht University

Exam "Classical Field Theory"

Tuesday, April 16th, 2013

Duration of the exam: 3 hours

1. *Use a seperate sheet for every excercise.*
2. *Write your name and initials on all sheets, on the first sheet also your address and your student ID number.*
3. *Write clearly, unreadable work cannot be corrected.*
4. *You may use your lecture notes.*

Consider a system of particles with mass m on some three-dimensional lattice with lattice constant a. The nearest-neighbor and next-nearest-neighbor particles are connected to each other with a spring with spring constant C. The displacement of the particles from their equilibrium position on the lattice is given by the vector field $u_i(\mathbf{x}, t)$, where $i = 1, 2, 3$ denote the x, y and z directions, respectively, and $\mathbf{x} = (x, y, z) \equiv (x_1, x_2, x_3)$.

a) In the continuum limit the action for this elastic medium is given by

$$S[\mathbf{u}] = \frac{1}{2}\int dt \int d\mathbf{x} \left\{ \rho \left(\frac{\partial u_i}{\partial t} \right)^2 - \mu \left(u_{ii} \right)^2 - \lambda \left(u_{ij} \right)^2 \right\} , \tag{1}$$

where the so-called strain tensor $u_{ij} \equiv (\partial_i u_j + \partial_j u_i)/2$ and summation over repeated indices is implied. Express ρ in terms of m, a. Argue that the elastic constants μ and λ are both proportional to C/a.

b) Derive the equation of motion for $u_i(\mathbf{x}, t)$ and give the dispersions $\omega_i(\mathbf{k})$ of the plane-wave solutions to this equation of motion. Sketch your results also in a figure in the $\omega - k$ plane, where k is the magnitude of the wave vector $\mathbf{k}$.

c) Give the Hamiltonian of the field theory in equation (1). Give also the Hamilton equations of motion that follow from the hamiltonian density and show that they are equivalent to the results obtained in b).

d) What term do we have to add to the continuum action if all the individual particles are attached to their equilibrium positions by a spring with spring constant C'? Determine the dispersions $\omega_i(\mathbf{k})$ of the plane-wave solutions in this case and sketch your results again in a figure in the $\omega - k$ plane.

e) The original action in equation (1) is invariant under the transformation $u_i(\mathbf{x}, t) \to u_i(\mathbf{x}, t) + \epsilon_i$. Derive the density and the current density associated with this symmetry. What is the corresponding conserved Noether charge? Give your physical interpretation of this conserved quantity.

f) Discuss finally the translational invariance $x_i \to x_i + \epsilon_i$ of the action in equation (1). Derive the conservation law associated with this symmetry. Compare your result for the Noether charge with your findings in e).