Classical field theory 2016 (NS-364B) – Final exam

Tue April 19 2016, 13:30-16:30 Gamma. In total 8 points = 80%
You have *three* hours to solve this exam. The exam is closed books.

Problem 1. Theoretical questions. *(2 points)*
Answer the questions below by a few sentences, a sketch, a formula or the like.

(A) *(0.5 points)* Consider Maxwell theory. What is a gauge transformation and how does it affect the equation of motion for the electric and magnetic field?

(B) *(0.5 points)* How many free parameters does the Lorentz group possess in 3+1 dimensional Minkowski space? What is the physical meaning of these parameters?

(C) *(0.5 points)* Write down the continuity equation for the electric charge density. Show that it implies the conservation of the total charge Q in a volume V provided there is no electric current through the boundary of V.

(D) *(0.5 points)* The mass diffusion equation is given by $\partial\rho/\partial t = D\nabla^2\rho$, with ρ the mass density, and D the diffusion constant. Briefly describe the steps to arrive at this equation, and discuss range of its validity.

Problem 2. The Euler-Lagrange and Hamilton's equations, Noether's theorem. *(3 points)*
Consider a system of N coupled real scalar fields whose action is,

$$S = \int d^4x \left[\frac{1}{2}G_{ab}(\vec{\phi})\partial_\mu\phi^a\partial_\nu\phi^b\eta^{\mu\nu}\right] , \tag{1}$$

where $a, b = 1, ..., N$, $\vec{\phi} = (\phi_1, ..., \phi_N)^T$. G_{ab} is known as a configurational metric (metric in the space of fields), and its inverse G^{ab} is given by $G^{ab}G_{bc} = \delta^a_{\ c}$. $\eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ is the inverse Minkowski metric.

(A) *(0.5 points)* Show that the Euler-Lagrange equations of motion are,

$$\partial_t\left[G_{ab}(\vec{\phi})\partial_t\phi^b\right] - \partial_i\left[G_{ab}(\vec{\phi})\partial_i\phi^b\right] = 0 . \tag{2}$$

(B) *(1 point)* Show that the Hamiltonian density is given by,

$$\mathcal{H} = \frac{1}{2}G^{ab}(\vec{\phi})\pi_a\pi_b + \frac{1}{2}G_{ab}\partial_i\phi^a\partial_i\phi^b . \tag{3}$$

and find Hamilton's equations of motion.

(C) *(1.5 points)* Assume that $G_{ab} = \delta_{ab}f(\vec{\phi}^{\,2})$, where f is some function of $\vec{\phi}^{\,2} = \phi_1^2 + ... + \phi_N^2$. In this case the Lagrangian density is invariant under rotations in the field space, $\vec{\phi} \to \mathcal{R}\vec{\phi}$, where $\mathcal{R}$ are orthogonal matrices satisfying $\mathcal{R}^T \cdot \mathcal{R} = \mathbb{1} = \mathcal{R} \cdot \mathcal{R}^T$.
Consider the case when $N = 2$. In this case the rotational matrix is given by,

$$\mathcal{R} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} , \tag{4}$$

where $\theta \in [0, 2\pi)$ is a rotational angle.

(i) Show that an infinitesimal version of that transformation is $\delta\phi^a = \theta\epsilon^a{}_b\phi^b$, where $\epsilon^a_{\ b}$ is an antisymmetric matrix whose entries are 1 and -1,

$$\epsilon^a{}_b = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \tag{5}$$

(ii) Show that the corresponding Noether current is,

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)}\delta\phi^a = f(\vec{\phi}^2)\left[(\partial^\mu\phi^{(1)})\phi^{(2)} - (\partial^\mu\phi^{(2)})\phi^{(1)}\right] . \tag{6}$$

(iii) Show that this current is indeed conserved,

$$\partial_\mu J^\mu = 0 . \tag{7}$$

Problem 3. Proca Lagrangian *(3 points)*
Consider the Proca Lagrangian which describes a massive vector field

$$\mathcal{L}_{\text{Proca}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{m^2}{2}A_\mu A^\mu \tag{8}$$

where A_μ is the vector potential, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor, and parameter m is the Compton wave number which is related to the mass of the field as $m = m_\gamma c/\hbar$.

(A) *(1 point)* Use the Euler-Lagrange equation to show that the Proca equations of motion are given by

$$\partial_\beta F^{\beta\alpha} - m^2 A^\alpha = 0 . \tag{9}$$

(B) *(1 point)* Calculate the stress energy-momentum tensor

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\lambda)}\partial^\nu A_\lambda - \eta^{\mu\nu}\mathcal{L} , \tag{10}$$

and write it in the following form

$$T^{\mu\nu} = \Theta^{\mu\nu} + \partial_\lambda S^{\lambda\mu\nu}, \tag{11}$$

where

$$\Theta^{\mu\nu} = -\left[F^{\mu\lambda}F^\nu{}_\lambda - \frac{1}{4}\eta^{\mu\nu}F_{\lambda\rho}F^{\lambda\rho} + m^2\left(A^\mu A^\nu - \frac{1}{2}\eta^{\mu\nu}A_\lambda A^\lambda\right)\right] \tag{12}$$

is the symmetric stress energy momentum tensor, and

$$S^{\lambda\mu\nu} = F^{\lambda\mu}A^\nu \tag{13}$$

is antisymmetric in the first two indices.

(C) *(1 point)* Show that the differential conservation law, in the Lorentz gauge $\partial_\mu A^\mu = 0$, takes the following form

$$\partial_\mu \Theta^{\mu\nu} = 0. \tag{14}$$

In doing so, you will need to make use of Bianchi identity $\partial^\rho F^{\nu\lambda} + \partial^\nu F^{\lambda\rho} + \partial^\lambda F^{\rho\nu} = 0$ and the Proca equation of motion.