Classical field theory 2016 (NS-364B) – Final exam

Tue April 19 2016, 13:30-16:30 Gamma. In total 8 points = 80%
You have *three* hours to solve this exam. The exam is closed books.

Problem 1. Theoretical questions. *(2 points)*
Answer the questions below by a few sentences, a sketch, a formula or the like.

(A) *(0.5 points)* Consider Maxwell theory. What is a gauge transformation and how does it affect the equation of motion for the electric and magnetic field?

(B) *(0.5 points)* How many free parameters does the Lorentz group possess in 3+1 dimensional Minkowski space? What is the physical meaning of these parameters?

(C) *(0.5 points)* Write down the continuity equation for the electric charge density. Show that it implies the conservation of the total charge Q in a volume V provided there is no electric current through the boundary of V.

(D) *(0.5 points)* The mass diffusion equation is given by $\partial\rho/\partial t = D\nabla^2\rho$, with ρ the mass density, and D the diffusion constant. Briefly describe the steps to arrive at this equation, and discuss range of its validity.

Problem 2. The Euler-Lagrange and Hamilton's equations, Noether's theorem. *(3 points)*
Consider a system of N coupled real scalar fields whose action is,

$$S = \int d^4x \left[\frac{1}{2} G_{ab}(\vec{\phi})\partial_\mu\phi^a\partial_\nu\phi^b\eta^{\mu\nu}\right], \tag{1}$$

where $a, b = 1, ..., N$, $\vec{\phi} = (\phi_1, ..., \phi_N)^T$. G_{ab} is known as a configurational metric (metric in the space of fields), and its inverse G^{ab} is given by $G^{ab}G_{bc} = \delta^a_{\ c}$. $\eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ is the inverse Minkowski metric.

(A) *(0.5 points)* Show that the Euler-Lagrange equations of motion are,

$$\partial_t\left[G_{ab}(\vec{\phi})\partial_t\phi^b\right] - \partial_i\left[G_{ab}(\vec{\phi})\partial_i\phi^b\right] = 0\,. \tag{2}$$

(B) *(1 point)* Show that the Hamiltonian density is given by,

$$\mathcal{H} = \frac{1}{2}G^{ab}(\vec{\phi})\pi_a\pi_b + \frac{1}{2}G_{ab}\partial_i\phi^a\partial_i\phi^b\,. \tag{3}$$

and find Hamilton's equations of motion.

(C) *(1.5 points)* Assume that $G_{ab} = \delta_{ab}f(\vec{\phi}^{\,2})$, where f is some function of $\vec{\phi}^{\,2} = \phi_1^2 + ... + \phi_N^2$. In this case the Lagrangian density is invariant under rotations in the field space, $\vec{\phi} \to \mathcal{R}\vec{\phi}$, where $\mathcal{R}$ are orthogonal matrices satisfying $\mathcal{R}^T\cdot\mathcal{R} = \mathbb{1} = \mathcal{R}\cdot\mathcal{R}^T$.
Consider the case when $N = 2$. In this case the rotational matrix is given by,

$$\mathcal{R} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \tag{4}$$

where $\theta \in [0, 2\pi)$ is a rotational angle.

(i) Show that an infinitesimal version of that transformation is $\delta\phi^a = \theta\epsilon^a_{\ b}\phi^b$, where $\epsilon^a_{\ b}$ is an antisymmetric matrix whose entries are 1 and -1,

$$\epsilon^a_{\ b} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \tag{5}$$

(ii) Show that the corresponding Noether current is,

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)}\delta\phi^a = f(\vec{\phi}^2)\left[(\partial^\mu\phi^{(1)})\phi^{(2)} - (\partial^\mu\phi^{(2)})\phi^{(1)}\right] . \tag{6}$$

(iii) Show that this current is indeed conserved,

$$\partial_\mu J^\mu = 0 . \tag{7}$$

Problem 3. Proca Lagrangian *(3 points)*
Consider the Proca Lagrangian which describes a massive vector field

$$\mathcal{L}_{\text{Proca}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{m^2}{2}A_\mu A^\mu \tag{8}$$

where A_μ is the vector potential, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor, and parameter m is the Compton wave number which is related to the mass of the field as $m = m_\gamma c/\hbar$.

(A) *(1 point)* Use the Euler-Lagrange equation to show that the Proca equations of motion are given by

$$\partial_\beta F^{\beta\alpha} - m^2 A^\alpha = 0 . \tag{9}$$

(B) *(1 point)* Calculate the stress energy-momentum tensor

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\lambda)}\partial^\nu A_\lambda - \eta^{\mu\nu}\mathcal{L} , \tag{10}$$

and write it in the following form

$$T^{\mu\nu} = \Theta^{\mu\nu} + \partial_\lambda S^{\lambda\mu\nu}, \tag{11}$$

where

$$\Theta^{\mu\nu} = -\left[F^{\mu\lambda}F^\nu_{\ \lambda} - \frac{1}{4}\eta^{\mu\nu}F_{\lambda\rho}F^{\lambda\rho} + m^2\left(A^\mu A^\nu - \frac{1}{2}\eta^{\mu\nu}A_\lambda A^\lambda\right)\right] \tag{12}$$

is the symmetric stress energy momentum tensor, and

$$S^{\lambda\mu\nu} = F^{\lambda\mu}A^\nu \tag{13}$$

is antisymmetric in the first two indices.

(C) *(1 point)* Show that the differential conservation law, in the Lorentz gauge $\partial_\mu A^\mu = 0$, takes the following form

$$\partial_\mu \Theta^{\mu\nu} = 0. \tag{14}$$

In doing so, you will need to make use of Bianchi identity $\partial^\rho F^{\nu\lambda} + \partial^\nu F^{\lambda\rho} + \partial^\lambda F^{\rho\nu} = 0$ and the Proca equation of motion.

Problem 1

(A) gauge transformation of electromagnetic potential

$$A^\mu \to A'^\mu = A^\mu + \partial^\mu \underbrace{\Lambda}_{\text{scalar field}}$$ 0.25

leaves $F^{\mu\nu}$ and thus $\vec{E}, \vec{B}$ unchanged

$\Rightarrow$ equations of motion also unchanged 0.25

(B) 6 free parameters 0.25

3 boosts (in 3 independent directions)

3 rotations (around 3 independent axes) } 0.25

(C) continuity equation

$$\frac{\partial}{\partial t} \rho + \vec{\nabla} \cdot \vec{j} = 0$$ 0.25

total charge in volume V

$$Q = \int_V d^3x \, \rho$$

conservation of Q

$$\frac{d}{dt} Q = \int_V d^3x \, \partial_t \rho = - \int_V d^3x \, \vec{\nabla} \cdot \vec{j} \overset{\text{Gauß theorem}}{=} - \int_{\partial V} d^2x \, \underbrace{\vec{n} \cdot \vec{j}}_{\text{current density through surface}} \overset{\vec{n}\cdot\vec{j}=0}{=} 0$$

∂V: surface of V

(D)

Conservation of mass: $\frac{\partial \rho}{\partial t} = -\nabla \cdot j$ 0.2

Fick's law $j = -D \nabla \rho$ 0.2

Validity: long wave length / small frequencies 0.1

PROBLEM 2 Solution

$$S = \int d^4x \Big[\underbrace{\tfrac{1}{2} G_{ab}(\vec{\phi})\, \partial_\mu \phi^a \partial_\nu \phi^b \eta^{\mu\nu}}_{\mathcal{L}} \Big]$$

$a, b = 1, \dots, N$; $\vec{\phi} = (\phi_1, \dots, \phi_N)^T$; $G^{ab} G_{bc} = \delta^a{}_c$

$\eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$

(A) EOM:

$$\frac{\delta S}{\delta \phi^a} = 0 = -\partial_\mu \left[G_{ab} \partial_\nu \phi^b \eta^{\mu\nu} \right] + \frac{1}{2} \frac{\partial G_{ab}}{\partial \phi^a} \partial_\mu \phi^a \partial_\nu \phi^b \eta^{\mu\nu}$$

$$\underbrace{\partial_\mu \left[G_{ab} \partial_\nu \phi^b \eta^{\mu\nu} \right]}_{\partial_t [G_{ab} \partial_t \phi^b] - \partial_i [G_{ab} \partial_i \phi^b]} = \frac{1}{2} \frac{\partial G_{ab}}{\partial \phi^a} \partial_\mu \phi^a \partial_\nu \phi^b \eta^{\mu\nu}$$

(B)

$$\Pi_a = \frac{\delta S}{\delta \partial_t \phi^a} = G_{ab} \partial_t \phi^b \Rightarrow \partial_t \phi^b = G^{ab} \Pi_b$$

$$\mathcal{H} = \Pi_a \partial_t \phi^a - \mathcal{L} = G^{ab} \Pi_a \Pi_b - \frac{1}{2} G_{ab} \partial_t \phi^a \partial_t \phi^b + \frac{1}{2} G_{ab} \partial_i \phi^a \partial_i \phi^b$$

$$\boxed{\mathcal{H} = \frac{1}{2} G^{ab} \Pi_a \Pi_b + \frac{1}{2} G_{ab} \partial_i \phi^a \partial_i \phi^b} \qquad H = \int d^4x\, \mathcal{H}$$

$$\dot{\Pi}_a = -\frac{\delta H}{\delta \phi^a} = \partial_i \left[G_{ab} \partial_i \phi^b \right] - \frac{1}{2} \frac{\partial G^{bc}}{\partial \phi^a} \Pi_b \Pi_c - \frac{1}{2} \frac{\partial G_{bc}}{\partial \phi^a} \partial_i \phi^b \partial_i \phi^c$$

$$\dot{\phi}_a = \frac{\delta H}{\delta \Pi_a} = G^{ab} \Pi_b$$

$$\partial [G^{ab} G_{bc}] = 0$$

EOM:

$$\partial_t \left[G_{ab} \partial_t \phi^b \right] - \partial_i \left[G_{ab} \partial_i \phi^b \right] = \frac{1}{2} \frac{\partial G_{ab}}{\partial \phi^a} \partial_\mu \phi^a \partial_\nu \phi^b$$

(c) $G_{ab} = \delta_{ab}\, f(\vec{\phi}^{\,2})$; $\vec{\phi}^{\,2} = \phi_1^2 + \phi_2^2$ $(N=2)$

i) $\vec{\phi} \to R\,\vec{\phi}$

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \to \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \simeq \begin{pmatrix}1 & \theta\\ -\theta & 1\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$$

$$\left.\begin{aligned}\delta\phi_1 &= \theta\,\phi_2\\ \delta\phi_2 &= -\theta\,\phi_1\end{aligned}\right\}\quad \boxed{\delta\phi^a = \theta\,\epsilon^a{}_b\,\phi^b}\ ;\quad \boxed{\epsilon^a{}_b = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}}$$

ii) $$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi^a)}\,\delta\phi^a = f(\vec{\phi}^{\,2})\,\delta_{ab}\,\partial^\mu\phi^b\,\delta\phi^a$$

$$j^\mu = f(\vec{\phi}^{\,2})\left[\partial^\mu\phi^{(1)}\delta\phi^{(1)} + \partial^\mu\phi^{(2)}\delta\phi^{(2)}\right] \qquad \phi^{(1)} \equiv \phi_1\ ;\ \phi^{(2)} \equiv \phi_2$$

$$\boxed{j^\mu = \theta\, f(\vec{\phi}^{\,2})\left[\partial^\mu\phi_1\,\phi_2 - \partial^\mu\phi_2\,\phi_1\right]} \qquad \theta = \text{arbitrary } (=1)$$

iii) $$\partial_\mu j^\mu = \partial_\mu\left\{f(\vec{\phi}^{\,2})\left[\partial^\mu\phi_1\,\phi_2 - \partial^\mu\phi_2\,\phi_1\right]\right\}$$

$$= \partial_\mu\left(f(\vec{\phi}^{\,2})\,\partial^\mu\phi_1\right)\phi_2 - \partial_\mu\left(f(\vec{\phi}^{\,2})\,\partial^\mu\phi_2\right)\phi_1 + f(\vec{\phi}^{\,2})\underbrace{\left[\partial^\mu\phi_1\,\partial_\mu\phi_2 - \partial^\mu\phi_2\,\partial_\mu\phi_1\right]}_{=0}$$

$$\overset{\text{EOM}}{=} \frac{1}{2}\frac{\partial f}{\partial\phi_1}\phi_2\,\partial_\mu\phi_1\,\partial^\mu\phi_2 - \frac{1}{2}\frac{\partial f}{\partial\phi_2}\phi_1\,\partial_\mu\phi_2\,\partial^\mu\phi_1$$

$\partial_\mu\phi_1\,\partial^\mu\phi_2$ ✓

$$= \frac{1}{2}\left(\frac{\partial f}{\partial\phi_1}\phi_2 - \frac{\partial f}{\partial\phi_2}\phi_1\right)\partial_\mu\phi_1\,\partial^\mu\phi_2 = \frac{1}{2}\frac{\partial f}{\partial\phi}\frac{1}{\phi}\underbrace{\left(\phi_1\phi_2 - \phi_2\phi_1\right)}_{=0}$$

$$\frac{\partial f}{\partial\phi_1} = \frac{\partial f}{\partial\phi}\frac{\partial\phi}{\partial\phi_1} = \frac{\partial f}{\partial\phi}\frac{\phi_1}{\phi}$$

$$\frac{\partial f}{\partial\phi_2} = \frac{\partial f}{\partial\phi}\frac{\partial\phi}{\partial\phi_2} = \frac{\partial f}{\partial\phi}\frac{\phi_2}{\phi}$$

$$\boxed{\partial_\mu j^\mu = 0}$$

$$\mathcal{L}_{Proca} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\frac{\partial\mathcal{L}}{\partial A_\alpha} = -\frac{1}{2}m^2\frac{\partial}{\partial A_\alpha}(A_\mu\,\eta^{\mu\nu}A_\nu) = -\frac{1}{2}m^2(\delta^\mu_\alpha\,\eta^{\mu\nu}A_\nu + \delta^\nu_\alpha\,\eta^{\mu\nu}A_\mu) = -m^2A^\alpha$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\alpha A_\beta)} = -\frac{1}{4}\eta^{\mu\rho}\eta^{\nu\sigma}\frac{\partial}{\partial(\partial_\alpha A_\beta)}\left[(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\rho A_\sigma - \partial_\sigma A_\rho)\right] =$$

$$= -\frac{1}{4}\eta^{\mu\rho}\eta^{\nu\sigma}\left(\delta^\mu_\alpha\delta^\nu_\beta F_{\rho\sigma} - \delta^\nu_\alpha\delta^\mu_\beta F_{\rho\sigma} + \delta^\rho_\alpha\delta^\sigma_\beta F_{\mu\nu} - \delta^\sigma_\alpha\delta^\rho_\beta F_{\mu\nu}\right) =$$

$$= -\frac{1}{4}(F^{\alpha\beta} - F^{\beta\alpha} + F^{\alpha\beta} - F^{\beta\alpha}) = -F^{\alpha\beta}$$

$$\partial_\beta\frac{\partial\mathcal{L}}{\partial(\partial_\beta A_\alpha)} - \frac{\partial\mathcal{L}}{\partial A_\alpha} = 0$$

$$\partial_\beta F^{\beta\alpha} - m^2A^\alpha = 0$$

$$T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\lambda)}\partial^\nu A_\lambda - \eta^{\mu\nu}\mathcal{L} =$$

$$\partial^\nu A_\lambda = \eta_{\lambda\alpha}\partial^\nu A^\alpha = \eta_{\lambda\alpha}(F^{\nu\alpha} + \partial^\alpha A^\nu) = F^\nu{}_\lambda + \partial_\lambda A^\nu$$

$$= -F^{\mu\lambda}\partial^\nu A_\lambda - \eta^{\mu\nu}\mathcal{L} =$$

$$= -F^{\mu\lambda}(F^\nu{}_\lambda + \partial_\lambda A^\nu) + \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} + \frac{1}{2}\eta^{\mu\nu}m^2A_\alpha A^\alpha =$$

$$= -F^{\mu\lambda}F^\nu{}_\lambda + \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} + \frac{1}{2}\eta^{\mu\nu}m^2A_\alpha A^\alpha - F^{\mu\lambda}\partial_\lambda A^\nu =$$

$$F^{\mu\lambda}\partial_\lambda A^\nu = \partial_\lambda(F^{\mu\lambda}A^\nu) - \underbrace{(\partial_\lambda F^{\mu\lambda})}A^\nu$$

$$\partial_\lambda F^{\mu\lambda} = -\partial_\lambda F^{\lambda\mu} \overset{e.o.m.}{=} -m^2A^\mu$$

$$= -F^{\mu\lambda}F^\nu{}_\lambda + \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} + \frac{1}{2}\eta^{\mu\nu}A_\alpha A^\alpha m^2 - \partial_\lambda(F^{\mu\lambda}A^\nu) - m^2A^\mu A^\nu =$$

$$= -F^{\mu\lambda}F^\nu{}_\lambda + \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} - m^2\left(A^\mu A^\nu - \frac{1}{2}\eta^{\mu\nu}A_\alpha A^\alpha\right) - \partial_\lambda\underbrace{(F^{\mu\lambda}A^\nu)}_{-F^{\lambda\mu}A^\nu} =$$

$$= \Theta^{\mu\nu} + \partial_\lambda S^{\lambda\mu\nu}$$

$$\partial_\mu\Theta^{\mu\nu} = -(\partial_\mu F^{\mu\lambda})F^\nu{}_\lambda - F^{\mu\lambda}(\partial_\mu F^\nu{}_\lambda) + \frac{1}{2}\eta^{\mu\nu}F_{\alpha\beta}(\partial_\mu F^{\alpha\beta}) - m^2(\partial_\mu A^\mu)A^\nu - m^2A^\mu(\partial_\mu A^\nu) + \eta^{\mu\nu}m^2A^\alpha\partial_\mu \ldots$$

$$= -(\partial_\mu F^{\mu\lambda})F^\nu{}_\lambda - \eta^{\mu\alpha}\eta^{\lambda\beta}F_{\alpha\beta}(\partial_\mu F^\nu{}_\lambda) + \frac{1}{2}F_{\alpha\beta}(\partial^\nu F^{\alpha\beta}) - m^2A^\mu(\partial_\mu A^\nu) + m^2A^\alpha(\partial^\nu A_\alpha) =$$

$$= -(\partial_\mu F^{\mu\lambda})F^\nu{}_\lambda - F_{\alpha\beta}(\partial^\alpha F^{\nu\beta}) + \frac{1}{2}F_{\alpha\beta}(\partial^\nu F^{\alpha\beta}) - m^2A^\alpha(\partial_\alpha A^\nu) + m^2A^\alpha(\partial^\nu A_\alpha) =$$

$$= -(\partial_\mu F^{\mu\lambda})F^\nu{}_\lambda + \frac{1}{2}F_{\alpha\beta}(\partial^\nu F^{\alpha\beta} - 2\partial^\alpha F^{\nu\beta}) + m^2A^\alpha\underbrace{(\partial^\nu A_\alpha - \partial_\alpha A^\nu)}_{F^\nu{}_\alpha} =$$

$$= -\underbrace{(\partial_\mu F^{\mu\lambda})}_{m^2A^\lambda}F^\nu{}_\lambda + m^2A^\lambda F^\nu{}_\lambda - \frac{1}{2}F_{\alpha\beta}\underbrace{(\partial^\alpha F^{\nu\beta} - \partial^\beta F^{\alpha\nu} + \partial^\nu F^{\beta\alpha})}_{=0} = 0$$