INSTITUTE FOR THEORETICAL PHYSICS
UTRECHT UNIVERSITY

Final exam Classical Field Theory

(270 points)

Monday, 11 April, 2022, 13:30-16:30

1. *Write your name and initials on all sheets, on the first sheet also your student ID number.*
2. *Write clearly, unreadable work cannot be corrected.*
3. *Make each exercise on a separate sheet of paper.*
4. *Give the motivation, explanation, and calculations leading up to each answer and/or solution.*
5. *Do not spend a large amount of time on finding (small) calculational errors. If you suspect you have made such an error, point it out in words.*
6. *This is NOT an open-book exam: books and/or notes are not allowed.*

1. ϕ^4-THEORY (*160 POINTS, 20 POINTS FOR EACH PART*)

Consider the action for 1+1-dimensional ϕ^4-theory, given by

$$S[\phi] = \int dx \left[-\frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - U(\phi) \right] \equiv \int dx L \, ,$$

with the potential

$$U(\phi) = \frac{1}{4}\lambda \left(\phi^2 - \frac{m^2}{\lambda} \right)^2 \, .$$

a) Show that the equation of motion for ϕ is given by

$$\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} = m^2 \phi - \lambda \phi^3 \, .$$

b) Evaluate the energy-momentum tensor $T^{\mu\nu} = g^{\mu\nu} L - \frac{\partial L}{\partial(\partial_\mu \phi)} \partial^\nu \phi$ in terms of the field ϕ and its derivatives.

c) Show that the energy density is given by

$$\epsilon = \frac{1}{2c^2}\left(\frac{\partial \phi}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2 + U(\phi) \, .$$

d) Give the energy-current density j, that obeys $\partial \epsilon/(c\partial t) = -\partial j/\partial x$, in terms of the field ϕ.

e) Show that the total energy is conserved. (Assume that the boundary terms that you may encounter are zero.)

f) The solitary-wave solution

$$\phi_0(x) = \frac{m}{\sqrt{\lambda}} \tanh\left(\frac{mx}{\sqrt{2}}\right) ,$$

is a time-independent solution of the field equation (NB: you do not need to show this). Show that $\phi(x,t) = \phi_0((x-ut)/(\sqrt{1-u^2/c^2}))$ is a time-dependent solution.

g) Linearize the field-equation around the minimum $\phi_+ = m/\sqrt{\lambda}$ of the potential $U(\phi)$ by means of writing $\phi = \phi_+ + \delta\phi$, and show that

$$\frac{1}{c^2}\frac{\partial^2 \delta\phi}{\partial t^2} - \frac{\partial^2 \delta\phi}{\partial x^2} = -2m^2 \delta\phi \ .$$

h) Compute the dispersion relation that follows from this latter equation.

2. MAXWELL THEORY (*110 POINTS*)

a) (*20 points*) At the lectures you have learned that the equations for the vector potential $\mathbf{A}(\mathbf{x},t)$ and the scalar potential $\phi(\mathbf{x},t)$ obey

$$\begin{aligned} \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\mathbf{A}(\mathbf{x},t) &= \mu_0 \mathbf{J}(\mathbf{x},t) \ ; \\ \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\phi(\mathbf{x},t) &= \frac{\rho(\mathbf{x},t)}{\epsilon_0} \ . \end{aligned} \tag{1}$$

Introduce the Green's function that obeys

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) G(\mathbf{x}-\mathbf{x}';t-t') = \delta(\mathbf{x}-\mathbf{x}')\delta(t-t') \ ,$$

with $\delta(t)$ the Dirac delta function. Show that the solution of Eqs. (1) for $\phi(\mathbf{x},t)$ is given by

$$\phi(\mathbf{x},t) = \int dt' \int d\mathbf{x}' G(\mathbf{x}-\mathbf{x}';t-t')\frac{\rho(\mathbf{x}',t')}{\epsilon_0} \ .$$

Analogously, we have that (you do *not* need to show this)

$$\mathbf{A}(\mathbf{x},t) = \int dt' \int d\mathbf{x}' G(\mathbf{x}-\mathbf{x}';t-t')\mu_0 \mathbf{J}(\mathbf{x},t) \ .$$

b) (*20 points*) Consider now a moving point charge, so that

$$\begin{aligned} \rho(\mathbf{x},t) &= q\delta(\mathbf{x}-\mathbf{r}(t)) \ ; \\ \mathbf{J}(\mathbf{x},t) &= q\mathbf{v}(t)\delta(\mathbf{x}-\mathbf{r}(t)) \ , \end{aligned} \tag{2}$$

with $\mathbf{v}(t) = d\mathbf{r}(t)/dt$. At the lecture you have learned that the appropriate Green's function is

$$G(\mathbf{x}-\mathbf{x}';t-t') = \theta(t-t')\frac{c}{4\pi|\mathbf{x}-\mathbf{x}'|}\delta(|\mathbf{x}-\mathbf{x}'| - c(t-t')) \ .$$

with $\theta(t)$ the Heaviside step function. Show that

$$\phi(\mathbf{x},t) = \int d\mathbf{x}' \frac{q}{4\pi\epsilon_0|\mathbf{x}-\mathbf{x}'|}\delta\left(\mathbf{x}' - \mathbf{r}(t - |\mathbf{x}-\mathbf{x}'|/c)\right) \ ,$$

and show that in the non-relativistic limit this reduces to

$$\phi(\mathbf{x},t) = \frac{q}{4\pi\epsilon_0|\mathbf{x}-\mathbf{r}(t)|} \ .$$

c) (*20 points*) In the remainder of this exercises we use relativistic notation for time and position, and frequency and wave vector, so that we have four-vectors with

$$x = \begin{pmatrix} ct \\ \mathbf{x} \end{pmatrix} ,$$

and

$$k = \begin{pmatrix} \omega/c \\ \mathbf{k} \end{pmatrix} ,$$

with components x^μ and k^μ, respectively. In the lecture you have also learned that the Maxwell equations can be derived from the action

$$S[A^\mu] = \int dx \left(-\frac{1}{4\mu_0 c} F_{\mu\nu} F^{\mu\nu} + \frac{1}{c} J_\mu A^\mu \right) , \tag{3}$$

with $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. Varying the above action yields (you do *not* need to show this)

$$\partial_\mu F^{\nu\mu} = \mu_0 J^\nu .$$

Perform a gauge transformation $A^\mu \to A^\mu + \partial^\mu \Lambda$, with Λ an arbitrary scalar function, and show that the action $S[A^\mu]$ in Eq. (3) is invariant under this transformation provided the current J^μ is conserved.

d) (*20 points*) We now add an extra term to the action so that

$$S[A^\mu] = \int dx \left(-\frac{1}{4\mu_0 c} F_{\mu\nu} F^{\mu\nu} + \frac{1}{c} J_\mu A^\mu - \frac{\lambda^2}{2} (\partial_\mu A^\mu)^2 \right) ,$$

with λ a constant. Show that the equations of motion become

$$\partial_\mu F^{\nu\mu} - \mu_0 c \lambda^2 \partial_\mu \partial^\nu A^\mu = \mu_0 J^\nu .$$

e) (*30 points*) Show that this equation for A^μ is solved by introducing the Green's function $G^\mu{}_\nu(x - x')$, which is given by

$$G^\mu{}_\nu(x - x') = \int \frac{dk}{(2\pi)^4} \left[k^2 \delta^\mu_\nu - (1 - \mu_0 c \lambda^2) k^\mu k_\nu \right]^{-1} e^{ik_\alpha (x^\alpha - x'^\alpha)} .$$