Solutions Exercise 1 final exam

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1a.

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \frac{\partial \mathcal{L}}{\partial \phi} \quad 5pt$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = -\partial^\mu \phi \qquad \frac{\partial \mathcal{L}}{\partial \phi} = -\frac{\partial \mathcal{U}}{\partial \phi} = -\frac{1}{2}\lambda\left(\phi^2 - \frac{m^2}{\lambda}\right)2\phi$$

$$= -\lambda \phi^3 + m^2 \phi \quad 5pt$$

$$\Rightarrow -\partial_\mu \partial^\mu \phi = -\lambda \phi^3 + m^2 \phi = -\frac{\partial \mathcal{U}}{\partial \phi} \quad 5pt$$

$$\Rightarrow \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} = -\lambda \phi^3 + m^2 \phi \quad 5pt$$

1b)

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = -\partial^\mu \phi \quad 5$$

$$\Rightarrow \quad T^{\mu\nu} = g^{\mu\nu}\left(-\frac{\partial_\alpha \phi\, \partial^\alpha \phi}{2} - \mathcal{U}(\phi)\right) \quad 15$$
$$+ \partial^\mu \phi\, \partial^\nu \phi$$

1c)

$$\varepsilon = T^{00} = \overset{g^{00} = -1}{-}c\left(-\frac{\partial_\alpha \phi\, \partial^\alpha \phi}{2} - \mathcal{U}(\phi)\right)$$
$$+ c\,\partial^0 \phi\, \partial^0 \phi \quad 5pt$$

$$= \frac{1}{2}(\partial_\alpha \phi)(\partial^\alpha \phi) + \mathcal{U}(\phi) + \partial^0 \phi\, \partial^0 \phi \quad 5pt$$

$$= -\frac{1}{2c^2}\left(\frac{\partial \phi}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2 + \frac{1}{c^2}\left(\frac{\partial \phi}{\partial t}\right)^2 + \mathcal{U}(\phi) \quad 5pt$$

$$= \frac{1}{2c^2}\left(\frac{\partial \phi}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2 + \mathcal{U}(\phi) \quad 5pt$$

$\longrightarrow$ because ∂

$$-\ \frac{1}{2c^2}\left(\frac{\partial}{\partial t}\right)^2 \ \frac{1}{2}\left(\frac{\partial}{\partial x}\right)^2\ \ u(\phi)$$

because $\partial^\mu = \frac{\partial}{\partial x_\mu} = \frac{+1}{c}\frac{\partial}{\partial t}$

1d) $$j = +T^{01} = +T^{10} = -\frac{1}{c}\left(\frac{\partial\phi}{\partial t}\right)\left(\frac{\partial\phi}{\partial x}\right)$$

10 pt   10 pt

CHECK (not necessary)

$$\frac{1}{c}\frac{\partial\varepsilon}{\partial t} = \frac{1}{c^2}\left(\frac{\partial\phi}{\partial t}\right)\left(\frac{\partial^2\phi}{\partial t^2}\right) + c^{-1}\left(\frac{\partial\phi}{\partial x}\right)\frac{\partial^2\phi}{\partial t\,\partial x} + c^{-1}\left(\frac{\partial u}{\partial\phi}\right)\frac{\partial\phi}{\partial t}$$

$$= \frac{1}{c}\left(\frac{\partial\phi}{\partial t}\right)\left(c^2\frac{\partial^2\phi}{\partial x^2} - c^2\frac{\partial u}{\partial\phi}\right) + c^{-1}\frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial t\,\partial x} + c^{-1}\left(\frac{\partial u}{\partial\phi}\right)\left(\frac{\partial\phi}{\partial t}\right)$$

$$= c^{-1}\frac{\partial\phi}{\partial t}\frac{\partial^2\phi}{\partial x} + c^{-1}\frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial t\,\partial x}$$

$$= c^{-1}\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial t}\right)\left(\frac{\partial\phi}{\partial x}\right) = -\frac{\partial j}{\partial x} \quad \text{ok!}$$

1e) $$\frac{\partial E}{\partial t} \overset{5}{=} \frac{\partial}{\partial t}\int dx\ \varepsilon \overset{5}{=} \int dx\ \frac{\partial\varepsilon}{\partial t} \overset{5}{=} -\int dx\ \frac{\partial j}{\partial x} \overset{5}{=} 0$$

(can also be done explicitly)

1f) $$\frac{\partial\phi}{\partial t} = \frac{\partial\phi_0}{\partial t}\left(x - ut/\gamma_u\right)$$

$$= \phi_0' \cdot -\frac{u}{\gamma_u} \qquad 5\text{ pt}$$

$$\frac{\partial^2\phi}{\partial t^2} = \phi_0'' \cdot \frac{u^2}{\gamma_u^2} \qquad 3\text{ pt}$$

$$\frac{\partial\phi}{\partial x} = \gamma_u\,\phi_0' \qquad 5\text{ pt}$$

$$\frac{\partial^2 \phi}{\partial x^2} = \gamma_u^2 \phi_0'' \quad 3pt$$

So: $$\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} = \left(\frac{u^2}{c^2}\gamma_u^2 - \gamma_u^2\right)\phi_0'' = -\frac{\partial U}{\partial \phi}(\phi_0)$$

$$= \underbrace{\gamma_u^2\left(\frac{u^2}{c^2} - 1\right)}_{-1}\phi_0'' = -\frac{\partial U}{\partial \phi}(\phi_0)$$

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$\rightarrow -\phi_0'' = -\frac{\partial U}{\partial \phi}(\phi_0)$ but since ϕ_0 obeys time-ind. eq. this is obeyed.

Hence, ϕ obeys time-dpt eq.

g) write $\phi = \phi_0 + \delta\phi$

then $$-\frac{\partial U}{\partial \phi} = \underbrace{-\frac{\partial U}{\partial \phi}(\phi_0)}_{0} - \frac{\partial^2 U}{\partial \phi^2}(\phi_0)\,\delta\phi + \dots$$

$$\overset{5}{=} \quad 0 \quad + m^2 - 3\lambda \phi^2\Big|_{\phi=\phi_0}$$

$$\overset{5}{=} -3\lambda \frac{m^2}{\lambda} + m^2 = -2m^2$$

$$\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = \frac{1}{c^2}\frac{\partial^2 \delta\phi}{\partial t^2} \quad \text{and} \quad \frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^2 \delta\phi}{\partial x^2} \quad 5$$

because ϕ_0 is time and position independent.

$$\rightarrow \quad \frac{1}{c^2}\frac{\partial^2 \delta\phi}{\partial t^2} - \frac{\partial^2 \delta\phi}{\partial x^2} \overset{5}{=} -2m^2\,\delta\phi$$

$$\Rightarrow \quad \frac{1}{c^2}\frac{\partial^2 \delta\phi}{\partial t^2} - \frac{\partial^2 \delta\phi}{\partial x^2} = -2m\,\delta\phi$$

b) Write $\delta\phi \overset{s}{=} A\, e^{ikx - i\omega t}$

$$\Rightarrow \quad \frac{1}{c^2}(-i\omega)^2 - (ik)^2 \overset{s}{=} -2m^2$$

$$\Rightarrow \quad -\frac{\omega^2}{c^2} + k^2 \overset{s}{=} -2m^2$$

$$\Rightarrow \quad \omega \overset{s}{=} \sqrt{c^2k^2 + 2m^2c^2}$$

Exercise 2

2a) Fill in $\phi(\vec{x},t) = \int d\vec{x}' \int dt' \, G(\vec{x}-\vec{x}';t-t') \dfrac{\rho(\vec{x}',t')}{\varepsilon_0}$

Then: $\left(\dfrac{1}{c^2}\dfrac{\partial^2}{\partial t^2} \nabla^2\right) \phi(\vec{x},t)$

$$\overset{5}{=} \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) \int d\vec{x}' \int dt' \, G(\vec{x}-\vec{x}';t-t') \frac{\rho(\vec{x}',t')}{\varepsilon_0}$$

$$\overset{5}{=} \int d\vec{x}' \int dt' \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) G(\vec{x}-\vec{x}';t-t') \frac{\rho(\vec{x}',t')}{\varepsilon_0}$$

$$\overset{5}{=} \int d\vec{x}' \int dt' \quad \delta(\vec{x}-\vec{x}')\,\delta(t-t') \, \frac{\rho(\vec{x}',t')}{\varepsilon_0}$$

$$\overset{5}{=} \frac{\rho(\vec{x},t)}{\varepsilon_0} \qquad \text{ok!}$$

2b) $\rho(\vec{x},t) = q\,\delta(\vec{x}-\vec{r}(t))$

So: $\phi(\vec{x},t) = \int dt' \int d\vec{x}' \; G(\vec{x}-\vec{x}';t-t') \, \dfrac{\rho(\vec{x}',t')}{\varepsilon_0}$

$$\overset{2}{=} \int dt' \int d\vec{x}' \quad \theta(t'-t') \quad \frac{cq}{4\pi\varepsilon_0 |\vec{x}-\vec{x}'|} \; \delta(|\vec{x}-\vec{x}'| - c(t-t'))$$
$$\delta(\vec{x}'-\vec{r}(t'))$$

put $t' = t - |\vec{x} - \vec{x}'|/c$ (NB $\theta(t^R - t^0) = 1$ then always!)

$$= \int d\vec{x}' \, \frac{cq}{4\pi\varepsilon_0 |\vec{x} - \vec{x}'|} \times \delta\left(\vec{x} - \vec{r}\left(t - \frac{|\vec{x} - \vec{x}'|}{c}\right)\right)$$

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Write $\vec{r}\left(t - \frac{|\vec{x} - \vec{x}'|}{c}\right) \simeq \vec{r}(t) - \frac{\partial \vec{r}}{\partial t} \frac{|\vec{x} - \vec{x}'|}{c}$

$$\simeq \vec{r}(t) - \frac{\vec{v}(t)}{c} |\vec{x} - \vec{x}'|$$

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Non-relativistic limit $\vec{v}(t) \ll c$ 5

$$\to \vec{r}\left(t - \frac{|\vec{x} - \vec{x}'|}{c}\right) \simeq \vec{r}(t)$$

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So $\phi(\vec{x}, t) \overset{2}{=} \int d\vec{x}' \, \frac{q}{4\pi\varepsilon_0 |\vec{x} - \vec{x}'|} \delta(\vec{x} - \vec{r}(t))$

$$\overset{2}{=} \frac{q}{4\pi\varepsilon_0 |\vec{x} - \vec{r}(t)|}$$

20c)

$$S[A^\mu] = \int dx \left(\frac{-1}{4\mu_0 c} (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) + \frac{1}{c} J_\mu A^\mu \right)$$

For $A^\mu \to A^\mu + \partial^\mu \Lambda$ we have

$$F^{\mu\nu} \to \partial^\mu (A^\nu + \partial^\nu \Lambda) - \partial^\nu (A^\mu + \partial^\mu \Lambda)$$

$$= F^{\mu\nu} + \underbrace{\partial^\mu \partial^\nu \Lambda - \partial^\nu \partial^\mu \Lambda}_{=0}$$ 5

And $$\int dx \, \frac{1}{c} J_\mu (A^\mu + \partial^\mu \Lambda) = \int dx \, \frac{1}{c} J_\mu A^\mu + \underbrace{\int dx \, \frac{1}{c} J_\mu \partial^\mu \Lambda}$$ 5

$$\int dx \, \frac{1}{c} (\partial^\mu J_\mu) \Lambda$$

So $$\delta S = \int dx \, \frac{1}{c} (\partial^\mu J_\mu) \Lambda = 0 \quad \text{if}$$

5 $\partial^\mu J_\mu = 0 \to$ conserved 5 current.

20d) Generally, the equations of motion are:

$$\frac{\partial \mathcal{L}}{\partial A_\nu} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) = 0$$

Here $\frac{\partial \mathcal{L}}{\partial A_\nu} = \frac{1}{c} J^\nu$ appears on right-hand side of eq. of motion with factor $\mu_0 c$.

So the new term of on the left-hand side is $\mu_0 c\, \partial_\mu \frac{\partial}{\partial(\partial_\mu A_\nu)}\left(-\frac{1}{2}^2 (\partial_\alpha A^\alpha)^2\right)$

Some argument leading up to this. +5 points

$\rightarrow$ $\mu_0 c \times \left(-\frac{1}{2}\right)^2 \partial_\mu \frac{\partial}{\partial(\partial_\mu A_\nu)} (\partial_\alpha A^\alpha)(\partial_{\alpha'} A^{\alpha'})$

First compute $\frac{\partial}{\partial(\partial_\mu A^\nu)} \underbrace{(\partial_\alpha A^\alpha)(\partial_{\alpha'} A^{\alpha'})}$

$$= 2\, \delta^\mu_\alpha \delta^\alpha_\nu\, \partial_{\alpha'} A^{\alpha'}$$

5 $= 2\, \delta^\mu_\nu\, \partial_{\alpha'} A^{\alpha'}$

So $\partial_\mu \frac{\partial}{\partial(\partial_\mu A^\nu)} = 2\, \partial_\mu \delta^\mu_\nu (\partial_{\alpha'} A^{\alpha'})$

5 $= 2\, \partial_\nu (\partial_{\alpha'} A^{\alpha'})$

And

$$-\lambda^2 \partial_\mu \frac{\partial}{\partial(\partial_\mu A_\nu)} = -\lambda^2 \partial^\nu(\partial_{\alpha'} A^{\alpha'})$$

$$= -\lambda^2 \partial^\nu \partial_\mu A^\mu$$

5

So: total eq. is:

$$\partial_\mu F^{\nu\mu} - \mu_0 c \lambda^2 \partial_\mu \partial^\nu A^\mu = \mu_0 J^\nu$$

e) Write out $F^{\nu\mu}$:

$$\partial_\mu(\partial^\nu A^\mu - \partial^\mu A^\nu) - \mu_0 c \lambda^2 \partial_\mu \partial^\nu A^\mu = \mu_0 J^\nu$$

Write:

$$\left[\partial_\mu \partial^\nu - (\partial_\alpha \partial^\alpha)\delta_\mu^\nu - \mu_0 c \lambda^2 \partial_\mu \partial^\nu\right] A^\mu = \mu_0 J^\nu$$

$$\Rightarrow \left[(1-\mu_0 c \lambda^2)\partial_\mu \partial^\nu - (\partial_\alpha \partial^\alpha)\delta_\mu^\nu\right] A^\mu(x) = \mu_0 J^\nu(x)$$

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Introduce Greens function which obeys:

$$\underbrace{\left[\left(1-\mu_0 c\,1^2\right)\partial_\mu\partial^\nu - (\partial_\alpha\partial^\alpha)\,\delta_\mu{}^\nu\right]}_{h_\mu{}^\nu} G^\mu{}_\nu(x-x') = \delta(x-x')$$

Then equation is solved by

$$A^\mu(x) = \int dx'\; G^\mu{}_\nu(x-x')\, J^\nu(x')$$

Because then

$$h_\mu{}^\nu A^\mu(x) = \int dx'\; h_\mu{}^\nu \cdot G^\mu{}_\nu(x-x')\, J^\nu(x')$$

$$= \int dx'\; \delta(x-x')\, J^\nu(x')$$

$$= J^\nu(x)$$

Write:

$$G^\mu{}_\nu(x-x') = \int \frac{dk}{(2\pi)^4}\; G^\mu{}_\nu(k)\, e^{ik_\alpha(x^\alpha - x'^\alpha)}$$

Then $\partial_\mu \to ik_\alpha$ in $h_\mu{}^\nu$ so

$$-\left[\left(1-\mu_0 c\,1^2\right)k_\mu k^\nu - k^2\delta_\mu{}^\nu\right] G^\mu{}_\nu(k) = 1$$

and

$$G^\mu{}_\nu(k) = -\left[\qquad \right]^{-1}$$