Final Exam- "Quantum Matter" (NS-371B)

July 1st 2013
Duration of the exam: 3 hours

1. Use a separate sheet for every exercise.

2. Write your name and initials in all sheets, on the first sheet also your address and your student ID number.

3. Write clearly, unreadable work cannot be corrected.

4. You are NOT allowed to use any kind of books or lecture notes. A list with some useful formulas is given at the end of the exam sheet.

1. Heat capacity of liquid ^{4}He

Liquid ^{4}He, which are bosonic atoms, becomes superfluid below a temperature of 2.17 K. Just as in a crystal, the low energy excitations of liquid ^{4}He are sound waves, whose quanta are the phonons. However, in liquid ^{4}He there is also another type of elementary excitation called the roton, see Figure 1. The dispersion relations for the phonons and rotons are, respectively,

$$\begin{aligned} \epsilon_p &= pc_1, \\ \epsilon_r &= \Delta + \frac{(p-p_0)^2}{2\mu_r}, \end{aligned}$$

where c_1 is the sound velocity, Δ is the energy gap of the rotons, μ_r is the effective mass of the rotons, and $p = \hbar k$ is the momentum of the atoms. As we have seen in crystals, the low energy excitations determine the behaviour of the thermodynamic quantities, such as the specific heat. In this sense liquid ^{4}He can be described as an ideal gas of phonons and an ideal gas of rotons.

1. (1.0) Let us first concentrate on the phonon gas. Calculate the temperature dependence of the free energy of the phonon gas, which is given by the free energy of the ideal Bose gas

$$F = -k_B T \int \ln(1+n) \frac{d^3p}{(2\pi\hbar)^3},$$

where $n = 1/(e^{\beta\epsilon} - 1)$ is the Planck distribution. Hint: write it as a dimensionless integral and extract the temperature dependence without solving the integral explicitly. Recall that d^3p is an infinitesimal volume element of a sphere.

2. (1.0) Calculate the temperature dependence of the entropy and specific heat of the phonon gas.

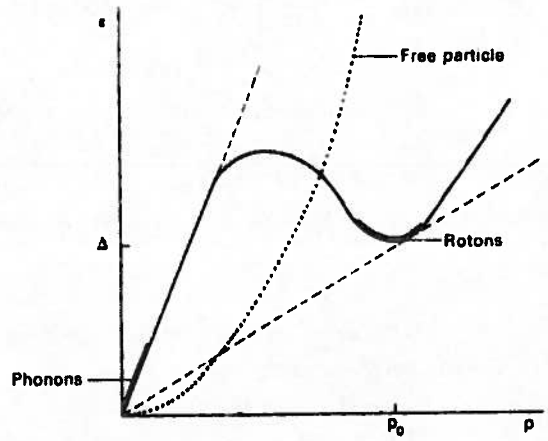

Figure 1: Spectrum of ^{4}He, with energy (vertical axis) versus momentum (horizontal axis), showing the phonon and roton excitations.

3. (1.0) Consider now the roton gas. Argue that if $\Delta \gg k_B T$ the Maxwell-Boltzmann distribution can be used for the rotons instead of the Bose-Einstein distribution function. Furthermore, show that the free energy of the ideal Bose gas reduces to the free energy of a Boltzmann gas
$$F \approx -k_B T \int n_B \frac{d^3p}{(2\pi\hbar)^3},$$
where n_B is the Boltzmann distribution.

4. (1.0) Calculate the temperature dependence of the free energy of the roton gas by approximating the integral assuming that $p_0 \gg \sqrt{\frac{\mu_r}{\beta}}$. Notice that the main contribution from the Gaussian term comes from $p \approx p_0$. Show that
$$F \approx -\frac{4\pi k_B T}{(2\pi\hbar)^3} e^{-\Delta/k_B T} p_0^2 \sqrt{2\pi \mu_r k_B T}.$$
As before, you could then determine the entropy and the specific heat, but you don't need to do it here.

2. Infinite Range Ising Model

We consider an Ising model in which *all* spins $\sigma_i = \pm 1$ interact with all other spins with interaction strength $-J/N$, where N is the number of spins in the system. In the presence of a magnetic field H, the partition function is given by

$$Z(\beta, H, N) = \sum_{\{\sigma_i\}} \exp\left\{\frac{\beta J}{2N}\sum_{i,j}\sigma_i\sigma_j + \beta H\sum_i \sigma_i\right\}, \tag{1}$$

where $\sum_{\{\sigma_i\}}$ denotes the sum over all of the 2^N spin configurations of the system.

(a) **(0.5)** Show that

$$\sum_{\{\sigma_i\}} e^{\beta(J\mu+H)\sum_i \sigma_i} = 2^N \prod_{i=1}^{N} \{\cosh\left[\beta(J\mu + H)\right]\}, \tag{2}$$

where μ is some auxiliary field which we need later.

(b) **(0.5)** Using the Gaussian identity,

$$\exp\left\{\frac{\beta J}{2N}\left(\sum_i \sigma_i\right)^2\right\} = \int_{-\infty}^{\infty}\frac{d\mu}{\sqrt{\frac{2\pi}{N\beta J}}}\exp\left\{-\frac{N\beta J}{2}\mu^2 + \beta J\mu\sum_{i=1}^{N}\sigma_i\right\}, \tag{3}$$

and equation (2), show that equation (1) can be written as

$$Z(\beta, H, N) = \int_{-\infty}^{\infty}\frac{d\mu}{\sqrt{\frac{2\pi}{N\beta J}}}\exp\left\{-\frac{N\beta J}{2}\mu^2 + N\log\left[2\cosh\left(\beta(H+\mu J)\right)\right]\right\}. \tag{4}$$

The magnetization m and susceptibility χ can be found from equation (4) by taking derivatives with respect to the magnetic field H.

(c) **(0.5)** The average magnetization is given as

$$\langle m\rangle = \frac{1}{N\beta}\frac{\partial \log Z}{\partial H} = \langle f(\mu, H, \beta)\rangle, \tag{5}$$

where the average is defined as

$$\langle \ldots \rangle = \frac{1}{Z}\int_{-\infty}^{\infty}\frac{d\mu}{\sqrt{\frac{2\pi}{N\beta J}}}\ldots\exp\left\{-\frac{N\beta J}{2}\mu^2 + N\log\left[2\cosh\left(\beta(J\mu+H)\right)\right]\right\}. \tag{6}$$

Calculate the function $f(\mu, H, \beta)$.

(d) **(0.5)** Calculate the susceptibility in the limit $N \to \infty$. The susceptibility is defined to be

$$\chi = \frac{1}{N}\frac{\partial\langle m\rangle}{\partial H}.$$

Show that it is given by

$$\chi = \beta \left[\left\langle \tanh^2(\beta(J\mu + H)) \right\rangle - \left\langle \tanh(\beta(J\mu + H)) \right\rangle^2 \right].$$

Hint: the limit $N \to \infty$ should be taken in the last step.

(e) **BONUS POINT (1.0)** Could you have expected the form you found for the magnetic susceptibility?

It is useful to define the function $\mathcal{F}(\mu, H)$,

$$\mathcal{F}(\mu, H) \equiv \frac{J}{2}\mu^2 - \frac{1}{\beta} \log\left[2\cosh(\beta(J\mu + H))\right], \tag{7}$$

such that

$$Z(\beta, H, N) = \int_{-\infty}^{\infty} \frac{d\mu}{\sqrt{\frac{2\pi}{N\beta J}}} e^{-\beta N \mathcal{F}(\mu, H)}. \tag{8}$$

In the thermodynamic limit, $N \to \infty$, the stationary phase method applied to Z becomes exact (for this model). The condition for stationarity is

$$\frac{\partial}{\partial\mu} \mathcal{F}(\mu, H) = 0. \tag{9}$$

The minima μ_i, where $i \in \{1, \ldots, n\}$ and n is the number of minima, found by the above condition give the dominant contributions to the partition function. Since for this model system the stationary phase method is exact in the thermodynamic limit $N \to \infty$, we can write the partition function as

$$Z(\beta, H, N) = \sum_{i=1}^{n} e^{-\beta N \mathcal{F}(\mu_i, H)}.$$

Note that we do not have to perform the integration over μ anymore! Moreover, if there is *one* absolute minimum μ_0, then in the thermodynamic limit the average magnetization will correspond to $m = \mu_0$. This means that the original auxiliary field μ, is actually the magnetization.

(f) **(0.5)** Show that the stationarity condition in equation (9) gives

$$\mu = \tanh\left[\beta(J\mu + H)\right] \tag{10}$$

(g) **(0.8)** Figure 2 shows the Landau free energy versus μ. What is the order of this phase transition? Justify your answer.

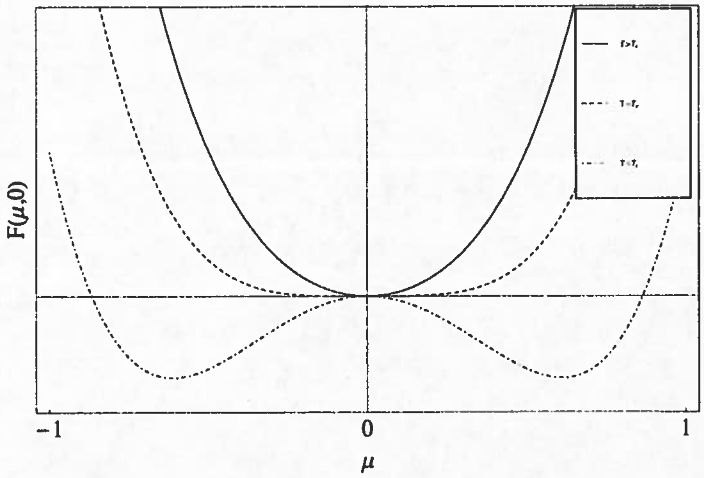

Figure 2: The Landau free energy versus μ for $H = 0$.

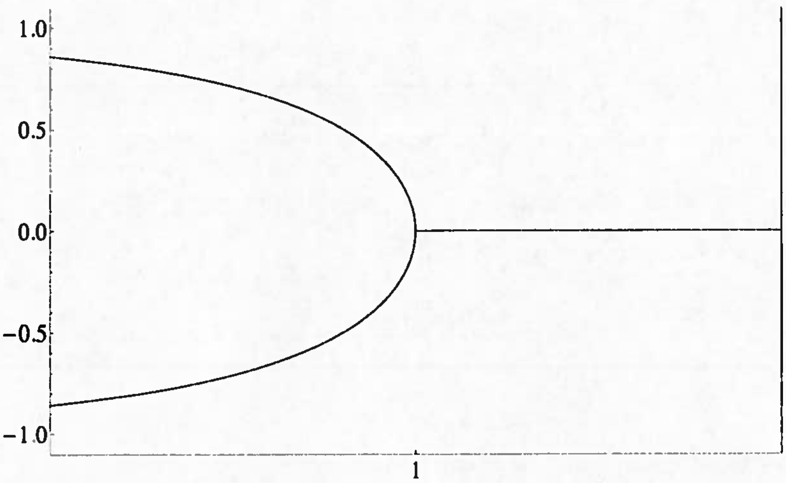

Figure 3: Something versus something...

(h) **(0.7)** We now want to calculate the susceptibility at zero field, $H = 0$, using the stationary phase method. Compute the susceptibility by differentiating equation (10) w.r.t. H, and show that it yields

$$\chi_0 = \frac{1}{\beta}\frac{\partial \mu}{\partial H}\bigg|_{H=0} = \frac{1-m^2}{1-\beta J(1-m^2)},$$

where the magnetization satisfies $m = \tanh(\beta J m)$. *Hint:* On both sides of equation (10) we have that the auxiliary field depends on the magnetic field, $\mu = \mu(H)$.

For $T > T_c$ there is only one minimum μ_0 which corresponds to zero magnetization $m = 0$. This state is called the paramagnetic phase. This means that the susceptibility is given by

$$\chi_0^{para} = \frac{k_B T}{k_B T - J},$$

(i) **(1.0)** Calculate the critical temperature for this phase transition at $H = 0$. Could you have anticipated the behavior of the susceptibility at T_c by looking at figure 2?

(j) **(0.5)** Knowing that figure 3 is in relation with figure 2, identify what is being plotted in the horizontal and vertical axis in figure 3. Explain your answer.

(k) **(0.5)** Close to the critical temperature T_c, we can expand $\tanh(x) \approx x - x^3/3$. Give the value of the critical exponent ν, which is defined to be

$$\mu \overset{T\to T_c}{\sim} \left(\frac{T_c - T}{T}\right)^{\nu}.$$

End

Formulas

- Maxwell-Boltzmann distribution: $g(\varepsilon) \propto \exp(-\varepsilon/k_B T)$
- Gibbs distribution:

$$P_i = \frac{\exp\{\beta(\mu N_i - E_i)\}}{\mathcal{Z}}, \qquad \mathcal{Z} = \sum_i \exp\{\beta(\mu N_i - E_i)\}$$

- For the different ensembles:

$$\Omega = e^{\beta T S}, \qquad Z = e^{-\beta F}, \qquad \mathcal{Z} = e^{-\beta \Phi_G},$$

where S is the entropy, T is the temperature, $\beta = (k_B T)^{-1}$, $F = -k_B T \ln Z$ is the Helmholtz function, and $\Phi_G = -k_B T \ln \mathcal{Z} = F - \mu N = -pV$ is the grand-potential.

- Recall the usual relations from quantum mechanics:

$$E = \hbar\omega, \qquad p = \hbar k, \qquad E = p^2/2m$$

- Photons:

$$\omega = ck, \qquad k = 2\pi/\lambda, \qquad c = \lambda\nu$$
$$E = \hbar kc = pc$$

- Planck distribution: $f(E) = \frac{1}{e^{\beta E}-1}$
- Fermi-Dirac- en Bose-Einstein distribution: $\ln \mathcal{Z} = \pm\sum_i \ln\left(1 \pm e^{\beta(\mu-E_i)}\right)$, with $f(E) = \frac{1}{e^{\beta(E-\mu)}\pm 1}$, where the sign $+$ stands for fermions and $-$ for bosons.
- Polylogarithm function $\mathrm{Li}_n(z)$:

$$\int_0^\infty dx \frac{x^{n-1}}{z^{-1}e^x \pm 1} = \mp\Gamma(n)\mathrm{Li}_n(\mp z), \qquad \mathrm{Li}_n(1) = \zeta(n),$$

where ζ is the Riemann zeta-function, and $\Gamma(n) = (n-1)!$ for n a positive integer.

$$\mathrm{Li}_1(z) = \ln\left(\frac{1}{1-z}\right)$$

- Gaussian integral: $\int_{-\infty}^{+\infty} dx e^{-ax^2} = \sqrt{\frac{\pi}{a}}$
- Stirling approximation: $\log(n!) \approx n\log n - n$