UTRECHT UNIVERSITY INSTITUTE FOR THEORETICAL PHYSICS

Quantum matter (Block 4, 2016/17)

DR. D. SCHURICHT
G. VAN MIERT, R. A. MULDER

Mid-term Exam (30 points)
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- Use a separate sheet for every exercise.
- Write your name on each sheet, on the first sheet also your student ID.
- Write clearly, unreadable work cannot be corrected.
- Give the motivation, explanation and calculation leading up to each answer and/or solution.
- Do not spend a large amount of time on finding (small) calculation errors. If you suspect you have made such an error, point it out in words.
- You have 90 minutes to work on the mid-term exam.

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1. Lattices and phonons (4+2+2+2 points = 10 points)

Consider a two-dimensional Bravais lattice defined by the lattice vectors $\vec{a}_1 = a\vec{e}_1$ and $\vec{a}_2 = a\vec{e}_2$, where a denotes the lattice constant. Furthermore, at each site of the Bravais lattice we put the three-site basis defined by the vectors $\vec{v}_1 = \vec{0}$, $\vec{v}_2 = \frac{a}{2}\vec{e}_1$ and $\vec{v}_3 = \frac{a}{2}(\vec{e}_1 + \vec{e}_2$. There are N lattice sites in the Bravais lattice.

(a) Sketch the lattice.

(b) How many acoustic and optical phonon modes do you expect?

(c) What is the specific heat at high temperatures?

(d) Determine the temperature dependence of the specific heat at low temperatures?

2. Ground-state properties of the Fermi gas (3+3+4 points = 10 points)

Consider N non-interacting electrons in a three-dimensional volume V. The system is at zero temperature, so we consider the ground state of the system.

(a) Show that the Fermi momentum, ie, the maximal momentum of the occupied state, is given by $k_{\mathrm{F}} = (3\pi^2 N/V)^{1/3}$.

(b) Calculate the ground-state energy E.

(c) Calculate the compressibility κ defined via $\kappa^{-1} = -V\partial P/\partial V$ from the pressure $P = -\partial E/\partial V$.

3. Bose–Einstein condensation (2+1+2+5 points = 10 points)

Consider bosonic atoms of mass m in a two-dimensional harmonic, ie, the Hamiltonian for each individual atom is given by

$$H = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \frac{m\omega_0^2}{2}(x^2 + y^2).$$

(a) Argue that the density of states depends linearly on the energy, ie, $D(\epsilon) \propto \epsilon$.

(b) Do you expect Bose–Einstein condensation to occur in this system? Why?

(c) How does the critical temperature T_0 at which Bose–Einstein condensation occurs depend on the particle density n, ie, determine the exponent α in $T \propto n^\alpha$?

(d) Determine the fraction $n_0(T)$ of particles in the condensate as a function of n, T and T_0. Sketch your result.