UTRECHT UNIVERSITY | INSTITUTE FOR THEORETICAL PHYSICS

Quantum matter (Block 4, 2014/15)

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Solutions: Retake exam (70 points)
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- Use a separate sheet for every exercise.
- Write your name on each sheet, on the first sheet also your student ID.
- Write clearly, unreadable work cannot be corrected.
- Except in exercise 1, give the motivation, explanation and calculations leading up to each answer and/or solution.
- Do not spend a large amount of time on finding (small) calculation errors. If you suspect you have made such an error, point it out in words.

1. Conceptual questions (3 points each = 15 points)

Answer the questions below by a few sentences, a sketch, a formula or the like.

(a) Sketch the two-dimensional triangular lattice and its Wigner–Seitz cell.

(b) How is the temperature dependence of the specific heat of a three-dimensional system of acoustic phonons at low and high temperatures?

(c) State the Bloch theorem.

(d) Briefly describe the qualitative difference between a paramagnet and a ferromagnet.

(e) Describe the difference between type-1 and type-2 superconductors.

2. Fermions in three dimensions (5+7 points = 12 points)

Consider a gas of non-interacting, spinless fermions in the three-dimensional volume $V = L^3$. The fermions possess the dispersion relation $E(\vec{p}) = c|\vec{p}|^\alpha$, $\vec{p} = (p_x, p_y, p_z)$, $\alpha > 0$. The chemical potential vanishes.

(a) Calculate the density of states.

Hint: Introduce spherical coordinates.

(b) Using this show that the specific heat satisfies $C \propto T^\delta$ and determine δ.

3. Boltzmann transport equation (6+7+7 points = 20 points)

Consider electrons with dispersion relation $E(\vec{k}) = \hbar^2\vec{k}^2/(2m)$ in a three-dimensional metal at temperature T. In the presence of an electric field $\vec{E}$ currents will flow. We describe this using the Boltzmann equation for the non-equilibrium distribution function

$g(\vec{k},t)$; the system is homogeneous so we can neglect the spatial dependence of g. In the relaxation-time approximation the Boltzmann equation reads

$$\left(\frac{\partial}{\partial t} - \frac{e\vec{E}}{\hbar} \cdot \vec{\nabla}_{\vec{k}}\right) g(\vec{k},t) = -\frac{1}{\tau}\Big[g(\vec{k},t) - f_0\big(E(\vec{k}) - \mu\big)\Big], \tag{1}$$

where e is the electron charge, τ the relaxation time, μ the chemical potential and $f_0(x) = (e^{x/(k_B T)} + 1)^{-1}$ the Fermi–Dirac distribution.

(a) Consider the steady state and show that the solution of (1) to linear order in $\vec{E}$ is given by

$$g(\vec{k},t) = f_0\big(E(\vec{k}) - \mu\big) + \frac{e\tau\hbar}{m} f_0'\big(E(\vec{k}) - \mu\big)\, \vec{E} \cdot \vec{k}, \tag{2}$$

where $f_0'(x) = \frac{\mathrm{d}}{\mathrm{d}x} f_0(x)$.

(b) Derive an integral representation for the energy conductivity σ_E defined via $\vec{j}_\mathrm{E} = \sigma_\mathrm{E}\vec{E}$ with the energy current density $\vec{j}_\mathrm{E}$.

(c) Show that if electrons were classical particles and τ is independent of temperature, the energy conductivity at $T \to 0$ becomes linear in the temperature, $\sigma_\mathrm{E} \sim T$.

4. Interacting spins (2+2+4+5+8+2 points = 23 points)

Consider two coupled spin-1/2's $\vec{S}_{1,2}$ subject to an Ising interaction and a magnetic field $\vec{B} = B\vec{e}_z$

$$H = J\, S_1^z\, S_2^z + \mu_0 B\big(S_1^z + S_2^z\big), \quad J > 0. \tag{3}$$

(a) On which Hilbert space does the Hamiltonian act? What is the Hilbert-space dimension?

(b) Which components of the total spin $\vec{S}_\mathrm{tot} = \vec{S}_1 + \vec{S}_2$ are conserved?

(c) Calculate the energies and eigenstates of the Hamiltonian as a function of J and B.

(d) Calculate the partition function Z, the free energy F and the magnetisation M. Use this to show that the susceptibility is given by $\chi = \hbar^2\mu_0^2/[k_B T(1 + e^{\hbar^2 J/(2k_B T)})]$.

(e) Derive the low- and high-temperature behaviour of the susceptibility obtained above. What defines the low- and high-temperature regimes? Sketch the susceptibility as a function of temperature.

(f) If one considers spin-1's instead, how does the susceptibility behave as $T \to 0$?