Quantum Field Theory 2010/11 – Final Exam, 3 Feb 2011, 14:00-16:00h

You must obtain at least 10 of the 20 points to pass the exam. The use of auxiliary materials such as books, notes, calculators, laptops etc. is *not* permitted. Please hand in all sheets you used for calculations.

Problem 1 - Tensor field dynamics (11 points)

Let the dynamics of the symmetric two-tensor field $f_{\mu\nu}(x)$ be described by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial^\lambda f_{\mu\nu})(\partial_\lambda f^{\mu\nu}) - \frac{1}{2}(\partial_\lambda f^\mu{}_\mu)(\partial^\lambda f^\nu{}_\nu) - (\partial^\lambda f^{\mu\nu})(\partial_\mu f_{\lambda\nu}) + (\partial_\lambda f^{\lambda\mu})(\partial_\mu f^\nu{}_\nu). \quad (1)$$

By symmetry, we mean $f_{\mu\nu}(x) = f_{\nu\mu}(x)$. As usual, indices on Lorentz tensors are raised and lowered using the Minkowski metric. One of your tasks is to find the (Feynman) propagator $D_{\kappa\lambda,\mu\nu}(x)$ in position space of the theory described by $\mathcal{L}$. Note that subproblem (c) below is independent of (a), (b).

(a) Trying to determine the propagator associated with (1), one encounters a problem similar to that in pure electrodynamics: after rewriting the Lagrangian (schematically) in the form $\mathcal{L} = \frac{1}{2} f \cdot M \cdot f$, the differential operator $M^{\kappa\lambda,\mu\nu}(x)$ is not invertible. Exhibit an explicit "zero-mode" of the Fourier transform $\tilde{M}$ of M, that is, some symmetric two-tensor $\tilde{f}^{(0)}_{\mu\nu}(k)$ with $\tilde{M}(k) \cdot \tilde{f}^{(0)}(k) \equiv \tilde{M}^{\kappa\lambda,\mu\nu}(k) \tilde{f}^{(0)}_{\mu\nu}(k) = 0$ to demonstrate the non-invertibility. [Hint: make sure to keep track correctly of the symmetry properties of M, D throughout.]

(b) Show that by adding a so-called gauge-fixing term $\mathcal{F}^\mu \mathcal{F}_\mu$ with a suitable prefactor c to $\mathcal{L}$ (which c?), where

$$\mathcal{F}_\mu = \partial^\nu f_{\mu\nu} - \frac{1}{2}\partial_\mu f_\nu{}^\nu, \quad (2)$$

the differential operator to be inverted becomes proportional to $\partial_\mu \partial^\mu \equiv \partial^2$. Invert this operator, paying attention to the fact that the identity operator I on the linear space of symmetric two-tensors is given by

$$D_{\mu\nu,\kappa\lambda}(D^{-1})^{\kappa\lambda,\rho\sigma} = I_{\mu\nu}{}^{\rho\sigma} := \frac{1}{2}(\delta_\mu{}^\rho \delta_\nu{}^\sigma + \delta_\mu{}^\sigma \delta_\nu{}^\rho), \quad (3)$$

and derive finally the propagator $D(x)$ of the gauge-fixed Lagrangian $\mathcal{L} + c\mathcal{F}^\mu \mathcal{F}_\mu$.

(c) The problem encountered in (a) above has to do with the presence of symmetry transformations on the fields which leave the action invariant (similar to what happens in electrodynamics). Show that $\mathcal{L}$ is invariant under the gauge transformations

$$f_{\mu\nu}(x) \to f_{\mu\nu}(x) + \partial_\mu \xi_\nu(x) + \partial_\nu \xi_\mu(x), \tag{4}$$

where the $\xi_\mu(x)$ are arbitrary functions on Minkowski space. (The ξ are supposed to be small in magnitude, which means that you should neglect terms which are of higher than linear order in ξ.) What are the conserved charges associated with the gauge transformations (4)?

Problem 2 - Divergences of scalar field theory (9 points)

Consider the real, massive scalar field theory with interaction term

$$\mathcal{L}_I = -\frac{g}{3!}\phi^3 - \frac{\lambda}{4!}\phi^4. \tag{5}$$

(a) Write down the complete classical action of the theory. What are the mass dimensions of the couplings g and λ if the theory lives in d-dimensional Minkowski space? In what dimensions is the theory (superficially) non-renormalizable?

(b) For the particular choice $d = 4$, what are the one-loop (amputated, momentum space) diagrams of the theory which are "one-particle irreducible"? (By definition, a one-particle irreducible diagram is one which cannot be split into two disconnected diagrams by cutting any one of its internal lines into two.) Which of these diagrams are associated with amplitudes that are ultraviolet divergent by power-counting?

(c) Regularizing any infinities by introducing a Lorentz-invariant momentum cut-off Λ, compute the leading, non-vanishing term in Λ of the amplitudes of those diagrams which you identified as power-counting divergent in (b). What are the symmetry weight factors for these diagrams? [You are allowed to make finite variable shifts in divergent momentum integrals. The identity

$$\frac{1}{AB} = \int_0^1 dx \, \frac{1}{(xA + (1-x)B)^2} \tag{6}$$

may come in handy.]