Quantum Field Theory 2011/12 – Final Exam, 2 Feb 2012, 14:00-16:00h

You must obtain at least 10 of the 20 points to pass the exam. The use of auxiliary materials such as books, notes, calculators, laptops etc. is *not* permitted. Please hand in all sheets you used for calculations.

Problem 1 - Vector field theory (12 points)

Consider the field theory with Lagrangian density

$$\mathcal{L}[B_\mu] = -\frac{1}{2\rho}(\partial^\mu B_\mu)^2 - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} \tag{1}$$

on four-dimensional Minkowski space, where the real tensor field $B_{\mu\nu}$ is given by $B_{\mu\nu} = \partial_\nu B_\mu - \partial_\mu B_\nu$ and ρ is a non-vanishing real constant. [Note that not all partial problems below depend on each other.]

(a) Compute the conjugate momenta Π^μ of the fields B_μ separately for time ($\mu = 0$) and spatial ($\mu = i$) components. Determine the Hamiltonian of the theory in terms of B_μ and Π^μ alone (i.e. with all explicit time derivatives eliminated) and simplify your final expression as much as possible.

(b) Compute the Euler-Lagrange equations following from (1). Show that for a specific choice of ρ (which one?) one obtains the wave equation. Write down the most general real classical solution for this case in terms of Fourier modes.

(c) Interpreting the vector field B_μ as a gauge potential, (1) resembles the Lagrangian of the photon field A_μ, which is invariant under $U(1)$-gauge transformations $A_\mu \to A_\mu + \partial_\mu\Lambda$. Is the Lagrangian (1) invariant under a similar transformation, and if yes, what is the associated conserved charge?

(d) Compute the propagator of the theory defined by (1) by inverting an appropriate differential operator extracted from $S = \int d^4x\, \mathcal{L}[B_\mu]$. [Hint: make a general algebraic ansatz for the propagator in momentum space, and then determine the free coefficients in the ansatz.]

(e) In the theory defined by (1), compute the force between two static sources, located at a spatial distance r from each other, which are coupled to the field B_μ. The sources are represented by an appropriately chosen external current $J^\mu(x)$, whose only non-vanishing component in this case is for $\mu = 0$.

For simplicity assume that the sources are point-like (delta functions) in space. What is the nature of the force and how does it depend on r? Just like in electromagnetism, it is convenient to introduce a small mass term for the field B_μ at an intermediate stage of the calculation. [Recall that the general procedure for determining the force is by first extracting an "interaction" energy E for the two sources by comparing the path integral (including J^μ) with $\exp(iET)$ for large time T. You may need the integral $\int_0^\infty dx\, \frac{x \sin ax}{x^2+b^2} = \frac{\pi}{2} e^{-ab}$, where $a > 0$ and $\mathrm{Re}(b) > 0$. Also, recalling the finite-dimensional integral $\int dq \exp\left(\frac{i}{2}aq^2 + ibq\right) \propto \exp\left(-\frac{i}{2}a^{-1}b^2\right)$ may help fix some minus signs and factors of i.]

Problem 2 - Interacting scalar field theory (8 points)

(a) Write down the classical action of a real massive scalar field theory with a $\lambda\phi^4$-interaction. What is the mass dimension of the coupling λ if the theory lives in d-dimensional Minkowski space? In what dimensions is the theory (superficially) non-renormalizable?

(b) For the particular choice $d = 4$, what are (the classes of) one-loop (amputated, momentum space) diagrams of the theory which are "one-particle irreducible"? (By definition, a one-particle irreducible diagram is one which cannot be split into two disconnected diagrams by cutting any one of its internal lines into two.) Which of these diagrams are associated with amplitudes that are ultraviolet divergent by power-counting?

(c) Regularizing any infinities by introducing a Lorentz-invariant momentum cut-off Λ, compute the leading, non-vanishing term in Λ of the amplitudes of those diagrams which you identified as power-counting divergent in (b). What are the symmetry weight factors for these diagrams? [You are allowed to make finite variable shifts in divergent momentum integrals. The identity

$$\frac{1}{AB} = \int_0^1 dx\, \frac{1}{(xA + (1-x)B)^2} \tag{2}$$

may come in handy.]