$\frac{\delta F}{\delta \pi}\frac{\delta G}{\delta \chi} - \frac{\delta G}{\delta \pi}\frac{\delta F}{\delta \chi}$

$F=\pi \quad G=\chi \; : \; 1 \qquad F=\chi \quad G=\chi \; : \; 0$

Quantum Field Theory 2011/12 – Retake Exam, 15 Mar 2012, 14:00-16:00h

You must obtain at least 10 of the 20 points to pass the exam. The use of auxiliary materials such as books, notes, calculators, laptops etc. is not permitted. Please hand in all sheets you used for calculations.

Problem 1 - Complex field theory (11.5 points)

Consider the classical field theory of a complex scalar field $\chi(x)$ with action

$$S = \int d^4x((\partial_\mu \chi^*)(\partial^\mu \chi) - m^2 \chi^* \chi). \tag{1}$$

on Minkowski space. Analyze this theory by treating $\chi(x)$ and $\chi^*(x)$ as the basic dynamical variables, rather than the real and imaginary parts of χ.

(a) Compute the conjugate momenta of the fields χ and χ^* and determine the Hamiltonian H of the theory as an integral over fields and momenta. What are the fundamental Poisson brackets among these fields (the classical analogues of the canonical commutation relations in the quantum theory)?

(b) Compute the classical Hamiltonian equations of motion for fields and momenta. By substituting these first-order equations into each other in an appropriate manner, derive second-order equations for χ and χ^*, and show that one obtains the Klein-Gordon equation.

(c) Show that the theory (1) is invariant under global phase transformation of the fields and compute the corresponding conserved current and charge Q. Express the charge as a function of the canonical field variables and momenta. $e^{i\xi}$

(d) Perform a canonical quantization of the fields and their conjugate momenta by expressing all of these quantum fields in terms of annihilation and creation operators. Make sure that the quantum fields are normalized in such a way that their canonical commutation relations are compatible with the standard commutation relations of the annihilation and creation operators (that is, $[\hat{a}(\vec{k}), \hat{a}^\dagger(\vec{k}')] = \delta^{(3)}(\vec{k}, \vec{k}')$ for each particle species.)

(e) Using (d), quantize the conserved charge Q to obtain an expression for $\hat{Q}$ in terms of annihilation and creation operators. Your final expression should contain only a single three-momentum integration. Fix any operator ordering ambiguities by choosing a normal ordering for $\hat{Q}$ (that is, with all annihilation operators to the right of all creation operators).

(f) What is the charge of the one-particle states of the theory, i.e. what is their eigenvalue with respect to $\hat{Q}$? What does this imply for the particle content of the quantum field theory corresponding to (1)?

Problem 2 - Dealing with spinors (8.5 points)

(a) Consider a massive Dirac spinor Ψ on four-dimensional Minkowski space. Establish that the upper two components (ψ_L) of Ψ transform under infinitesimal Lorentz transformations according to

$$\psi_L \to (\mathbf{1} - \frac{i}{2}\vec{\theta}\cdot\vec{\sigma} - \frac{1}{2}\vec{\beta}\cdot\vec{\sigma})\psi_L, \tag{2}$$

where θ_i are three rotation parameters, β_i are three boost parameters, and the σ_i denote the Pauli matrices. As usual, write a general Lorentz transformation on spinors in terms of the exponential of a linear combination of the generators of the Lorentz group in the spinor representation, using the γ-matrices (in the Weyl representation) and identify the coefficients in your ansatz appropriately with the θ_i and β_i. What is the transformation law analogous to (2) for the lower two components (ψ_R) of the Dirac spinor? Which relativistic wave equation does ψ_L satisfy for the case of a free field with vanishing mass?

(b) Define a new two-component field χ_a, $a = 1, 2$, which transforms like (ψ_L) under Lorentz transformations. One can write down a classical, relativistic equation for χ as a *massive* field, namely,

$$(i\partial_t - i\vec{\sigma}\cdot\vec{\nabla})\chi - im\sigma_2\chi^* = 0, \tag{3}$$

where the star denotes complex conjugation and $m > 0$. Show that if χ satisfies (3), each component of χ satisfies the Klein-Gordon equation. (The relativistic invariance of (3) is not immediately obvious, but we are not going to prove it here.)

(c) Write down the action of a field theory of fermions which is invariant under global chiral transformations (prove the invariance and any identities used explicitly). How do the individual fermion field components transform under a (finite, not infinitesimal) chiral transformation?

[Recall that the Pauli matrices, σ_i, $i = 1, 2, 3$ are given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4}$$

and satisfy $\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$. The Weyl representation of the γ-matrices is

$$\gamma^0 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \tag{5}$$

for $i = 1, 2, 3$.]