Instituut voor Theoretische Fysica, Universiteit Utrecht

FINAL EXAM QUANTUM FIELD THEORY

30 January, 2014

- The duration of the exam is 3 hours.
- The exam is closed-book.
- Usage of a calculator and a dictionary is allowed.
- Use different sheets for each exercise.
- Write clearly, unclear writing will not be evaluated.
- Write your name and initials on every sheet handed in.
- Divide your available time wisely over the exercises.

Problem 1 (15 points)

1. Show that for a vector field $A_\mu(x)$ the reality condition $A_\mu(x) = A^*_\mu(x)$ is Lorentz invariant. Show that the same reality condition for a spinor field $\psi(x)$ is not compatible with Lorentz invariance in a generic representation of γ-matrices.
2. Show that in the Dirac representation for γ-matrices the Majorana condition for a spinor $\psi(x) = \psi^c(x)$ with $\psi^c(x) = \gamma^2\psi^*(x)$ is Lorentz invariant and that $(\psi^c)^c = \psi$.
3. Show that in the Majorana representation for γ-matrices the Majorana condition for a spinor $\psi(x) = \psi^c(x)$ with $\psi^c(x) = \psi^*(x)$ is Lorentz invariant.
4. Find a unitary transformation which changes an overall sign of the matrix γ^2 only, keeping all remaining γ-matrices unchanged.

Problem 2 (10 points)

Suppose c_i, $i = 1,2$ are two fermionic annihilation operators and $c_i^\dagger$ the corresponding creation operators with the anti-commutation relations

$$\{c_i, c_j\} = \{c_i^\dagger, c_j^\dagger\} = 0\,, \qquad \{c_i, c_j^\dagger\} = \delta_{ij}\,.$$

These operators act on a fermionic Fock space $\mathcal{F}$ that is generated by the action of the creation operators from the vacuum $|0\rangle$ with $c_i|0\rangle = 0$, $i = 1, 2$.

Questions:

1. Show that the Fock space has dimension $\dim \mathcal{F} = 4$.

2. Define γ-operators via

$$\begin{aligned} \gamma^0 &= c_1 + c_1^\dagger \qquad \gamma^1 = i(c_2 + c_2^\dagger) \\ \gamma^2 &= c_1 - c_1^\dagger \qquad \gamma^3 = c_2 - c_2^\dagger\,. \end{aligned}$$

Using the anti-commutation relations between oscillators, show that these γ-operators satisfy the Dirac algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\,.$$

Problem 3 (30 points)

Two coupled harmonic oscillators are described by the hamiltonian

$$H = \tfrac{1}{2}(p_1^2 + \omega q_1^2) + \tfrac{1}{2}(p_2^2 + \omega q_2^2) + \lambda q_1 q_2\,,$$

where λ is a coupling constant. The vacuum-to-vacuum transition amplitude in the presence of sources is

$$Z[J_1, J_2] = N \int \mathscr{D}q_1 \mathscr{D}q_2 \exp\left(iS + i\int \mathrm{d}t\, J_1(t)q_1(t) + i\int \mathrm{d}t\, J_2(t)q_2(t)\right),$$

where S is the corresponding action and the normalization factor N is chosen such that $Z[0,0] = 1$.

Evaluate the correlation function $\langle q_1(t_1)q_2(t_2)\rangle$ at orders λ and λ^2 in perturbation theory in terms of the corresponding Feynman propagator.

Problem 4 (20 points)

Consider the scalar ϕ^3 theory governed by the Lagrangian

$$\mathscr{L} = \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{m^2}{2}\phi^2 - \tfrac{g}{3!}\phi^3\,.$$

1. Formulate the Feynman rules for the theory (in the x-space).
2. Show, by power counting, that the theory is renormalizable in $d = 6$ space-time dimensions.
   Hint. Use superficial degree of divergence.
3. Write an expression for the one-loop self-energy Σ in $d = 6$ in the momentum space (but do not evaluate it).

Problem 5 (25 points)

The transition amplitude for the harmonic oscillator has the form

$$W(q_1,t_1;q_2,t_2) = \left(\frac{m\omega}{2\pi\hbar|\sin\omega(t_1-t_2)|}\right)^{1/2} e^{\frac{im\omega}{2\hbar\sin\omega(t_1-t_2)}\left[(q_1^2+q_2^2)\cos\omega(t_1-t_2)-2q_1q_2\right]}.$$

Let W_E is an euclidean version of W obtained by Wick's rotation $t \to -i\tau$.

Questions:

1. Compute the "partition function" $(\tau_1 > \tau_2)$

$$Z = \int \mathrm{d}q \, W_E(q,\tau_1,-q,\tau_2)\,,$$

   where *anti-periodic* boundary condition for $q(t)$, *i.e.* $q_2(t_2) = -q_1(t_1)$, were assumed.

2. Show that the inverse of this partition function coincides with the partition function for a *fermionic harmonic oscillator* with the Hamiltonian

$$H = \frac{\hbar\omega}{2}\left(b^\dagger b - b b^\dagger\right),$$

   where the operators b and $b^\dagger$ obey the anti-commutation relations

$$\{b,b\} = 0\,, \qquad \{b^\dagger,b^\dagger\} = 0\,, \qquad \{b,b^\dagger\} = 1.$$

   Hint. Use the operator formalism to compute the partition function.

3. Show that the so-called Witten index W for the fermionic harmonic oscillator

$$W = \mathrm{tr}\left((-1)^{N_F} e^{-\beta H}\right),$$

   where N_F is a fermionic number operator, coincides with the inverse of the actual partition function of the bosonic harmonic oscillator.