Rumord

DEPARTEMENT NATUUR- EN STERRENKUNDE
UNIVERSITEIT UTRECHT

FINAL EXAM Quantum Field Theory - NS-TP401M

Thursday, January 29, 2015, 13:30-16:30, Olympos Hal 1.

1) Start every exercise on a separate sheet. Write on each sheet: your name and initials, and your studentnumber.

2) Please write legibly and clear. Unreadable handwriting cannot be marked!

3) The exam consists of **three** exercises and counts for 50% of the total final mark.

4) No lectures notes or any other material (books, calculators, ...) are allowed. A formularium instead is given.

Formularium

We use natural units, in which $c = \hbar = 1$ in this exam. The Minkowski metric in four spacetime dimensions is $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$. The Dirac matrices $(\gamma^\mu)_\alpha{}^\beta$ satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} . \tag{1}$$

The Delta function has an integral representation given by

$$\int_{-\infty}^{+\infty} \mathrm{d}p\, e^{ipx} = 2\pi\delta(x) . \tag{2}$$

1. Scalar field on a circle (4 points)

Consider the action of a real scalar field in two spacetime dimensions,

$$S = \int \mathrm{d}x\mathrm{d}t \left(-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \lambda\phi^4 \right) . \tag{3}$$

Assume that the spatial coordinate x parametrizes a circle of length $L = 2\pi R$, and decompose the scalar field in terms of Fourier modes

$$\phi(x,t) = \underbrace{\sum_{n=-\infty}^{+\infty}}_{\text{only time?}} \phi_n(t)\, e^{inx/R} . \tag{4}$$

i) Express the action in terms of these Fourier modes and show that you obtain a quantum-mechanical model of an infinite tower of harmonic oscillators $\phi_n(t)$, where $n = 0, \pm 1, \pm 2, \cdots$, with frequencies (masses)

$$M_n^2 = m^2 + \frac{n^2}{R^2} \, . \tag{5}$$

Rescale the ϕ_n such that the kinetic energy reads $\frac{1}{2}(\partial_t \phi_0)^2 + \sum_{n>0} |\partial_t \phi_n|^2$.

ii) Write down the propagators and vertices for this quantum-mechanical model.

iii) Draw the Feynman diagram(s) that contribute to the self-energy of ϕ_0 in the one-loop approximation, and write down the corresponding expression.

iv) Compute now explicitly the one-loop correction to the ϕ_0-mass by evaluating the loop integrals. Use the fact that the propagator,

$$\Delta(x-y) = \frac{1}{i(2\pi)^d} \int \mathrm{d}^d k \, \frac{e^{ik_\mu (x-y)^\mu}}{k^2 + M^2 - i\epsilon} \, , \tag{6}$$

for $d = 1$, is given by $\Delta(x-y) = \frac{1}{2M} e^{-iM|x-y|}$, and hence $\Delta(0) = \frac{1}{2M}$.

2. Green's function for a Dirac spinor (2 points)

Consider the propagator for a Dirac spinor in four spacetime dimensions

$$\Delta_\alpha{}^\beta(x) = \frac{1}{i(2\pi)^4} \int \mathrm{d}^4 p \, \frac{(-i\gamma^\mu p_\mu + m)_\alpha{}^\beta}{p^2 + m^2} \, e^{ip\,x} \, . \tag{7}$$

i) Show that it satisfies the equation for a Green's function,

$$(c\gamma^\mu \partial_\mu + m)\Delta = d\,\delta^4(x) \, , \tag{8}$$

for some coefficients c and d. Find these coefficients.

ii) How many physical degrees of freedom[redacted] does a Dirac spinor have and why ?

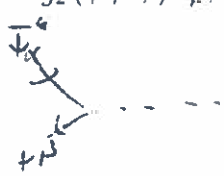
3. Renormalizability of Yukawa couplings ? (4 points)

Consider a field theory for a massive Dirac spinor ψ and a scalar field ϕ in $d = 4$ spacetime dimensions. The free Lagrangian (density) reads

$$\mathcal{L}_0 = -\bar{\psi}(\gamma^\mu \partial_\mu + M)\psi - \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 \, . \tag{9}$$

We consider two types of interactions between these fields,

$$\mathcal{L}_1 = g_1 \, (\bar{\psi}\psi) \, \phi \, , \qquad \mathcal{L}_2 = g_2 \, (\bar{\psi}\gamma^\mu \psi) \, \partial_\mu \phi \, . \tag{10}$$



i) Write down the Feynman rules for the two theories given by $\mathcal{L}_0 + \mathcal{L}_1$ and $\mathcal{L}_0 + \mathcal{L}_2$.

ii) Consider the one-loop diagrams contributing to the fermion self-energy and give the explicit expressions for the two theories. Are the diagrams divergent and, if so, indicate the counterterms that one needs to absorb all the (one-loop) divergencies.

iii) What are the mass dimensions of the fields and the coupling constants in the two theories ? Are these theories renormalizable by power counting and why (not) ?

iv) Consider now the case of two spacetime dimensions, $d = 2$, and answer question iii) again for $d = 2$.