DEPARTEMENT NATUUR- EN STERRENKUNDE
UNIVERSITEIT UTRECHT

FINAL EXAM Quantum Field Theory - NS-TP401M

Thursday, January 28, 2016, 17:00-20:00, Educatorium Alfa.

1) Start every exercise on a separate sheet. Write on each sheet: your name and initials, and your student number.
2) Please write legibly and clear. Unreadable handwriting cannot be marked!
3) The exam consists of **two** exercises.
4) No lectures notes or any other material (books, calculators, ...) are allowed. A formularium instead is given.

Formularium

We use natural units, in which $c = \hbar = 1$ in this exam, unless stated otherwise.

- The Minkowski metric in four spacetime dimensions is $\eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1)$.
- The Dirac matrices $(\gamma^\mu)_\alpha{}^\beta$ satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \; . \tag{1}$$

- The Delta function has an integral representation given by

$$\int_{-\infty}^{+\infty} \mathrm{d}p \, e^{ipx} = 2\pi\delta(x) \; . \tag{2}$$

- Notation:

$$J \cdot \phi \equiv \int \mathrm{d}^4x \, J(x)\phi(x) \; , \qquad (J, \Delta J) \equiv \int \mathrm{d}^4x \int \mathrm{d}^4y \, J(x)\Delta(x,y)J(y) \; . \tag{3}$$

1. Scalar Yukawa theory (4 points)

Consider two real scalar fields ψ and ϕ with action

$$S[\psi, \phi] = \int \mathrm{d}^4x \Big[-\frac{1}{2}\partial_\mu\psi\partial^\mu\psi - \frac{1}{2}m_\psi^2\psi^2 - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m_\phi^2\phi^2 - \lambda\phi^2\psi \Big] \; , \tag{4}$$

and coupling constant λ. We add two source terms to the action and define (see formularium for the notation)

$$\frac{i}{\hbar}S_J[\psi,\phi] \equiv \frac{i}{\hbar}S[\psi,\phi] + J_\psi \cdot \psi + J_\phi \cdot \phi\ , \tag{5}$$

and for the path integral

$$W_J \equiv \int \mathcal{D}\psi\mathcal{D}\phi\, e^{\frac{i}{\hbar}S_J[\psi,\phi]}\ . \tag{6}$$

i) Show that we can write

$$S[\psi,\phi] = \frac{1}{2}(\psi, A_\psi\,\psi) + \frac{1}{2}(\phi, A_\phi\,\phi) + S_I[\psi,\phi]\ , \tag{7}$$

where S_I represents the interactions. Find explicit expressions for the differential operators $A_\psi(x,y)$ and $A_\phi(x,y)$.

ii) Show that, up to an irrelevant normalization factor, the path integral can be written as

$$W_J = e^{\frac{i}{\hbar}S_I[\frac{\partial}{\partial J_\psi},\frac{\partial}{\partial J_\phi}]}e^{\frac{1}{2}(J_\psi,\Delta_\psi J_\psi)+\frac{1}{2}(J_\phi,\Delta_\phi J_\phi)}\ , \tag{8}$$

with $\Delta(x,y) = i\hbar A^{-1}(x,y)$.

iii) Expand the interaction term S_I to linear order in λ and take the derivatives with respect to the sources. Draw the corresponding Feynman diagrams and use different types of lines for the propagators for Δ_ψ and Δ_ϕ.

iv) Using Feynman diagrams, draw the one-loop diagrams for the self-energies for ψ and for ϕ. Write down the expressions that compute these diagrams. What is the degree of divergence of these diagrams ?

2. Massive vector fields (6 points)

Consider the following Lagrangian for a massive vector field coupled to a Dirac spinor in four spacetime dimensions,

$$\mathcal{L} = -\frac{1}{4}(\partial_\mu V_\nu - \partial_\nu V_\mu)^2 - \frac{1}{2}M^2 V_\mu V^\mu - \bar{\psi}(\gamma^\mu\partial_\mu + m)\psi + ieV_\mu\bar{\psi}\gamma^\mu\psi\ . \tag{9}$$

i) Show that for $M = 0$, the Lagrangian has a gauge symmetry.

ii) Write down the equations of motion for all the fields.

iii) Fourier transform the fields and determine the propagators for the massive vector field V_μ and the Dirac fermion.

iv) Consider the case $M = 0$. Determine the (mass) dimensions of the fields and coupling constants. Draw the one-loop Feynman diagram for the self-energy of the vector field and determine the degree of divergence from the momentum integrals. Is the theory renormalizable by power counting ? Explain your answer.

v) Now consider $M \neq 0$ and focus on the longitudinal mode of the vector field

$$V_\mu = \frac{1}{M}\partial_\mu \phi \,. \tag{10}$$

Rewrite the Lagrangian for this longitudinal mode and determine again the mass dimensions of the fields and coupling constants. Write down the Feynman rules for this model.

vi) Find out whether the theory is renormalizable by power counting. If not, write down Feynman diagrams that lead to divergences and counterterms that cannot be absorbed in the original Lagrangian.