Midterm Quantum Field Theory 2022

Instructions:

- Write your name and student number on every sheet.
- Make sure your answers are understandable and readable.
- Please solve each problem on a different sheet.

Formularium and conventions:

- We use the Minkowski metric in four spacetime dimensions $\eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1)$
- Klein-Gordon equation:
$$(-\partial_x^2 + m^2)\phi(x) = 0$$
- Complex scalar field:
$$\phi(x) = \int \mathrm{d}\tilde{k}(a(\vec{k})e^{ikx} + b^\dagger(\vec{k})e^{-ikx})$$
$$\phi^\dagger(x) = \int \mathrm{d}\tilde{k}(a^\dagger(\vec{k})e^{-ikx} + b(\vec{k})e^{ikx})$$
where $\mathrm{d}\tilde{k} = \frac{\mathrm{d}^3\vec{k}}{2\omega(\vec{k})(2\pi)^3}$ and $\omega = \sqrt{\vec{k}^2 + m^2}$.
- Non-vanishing, equal-time canonical commutation relations :
$$[\phi^\dagger(\vec{x}, t), \Pi^\dagger(\vec{y}, t)] = [\phi(\vec{x}, t), \Pi(\vec{y}, t)] = i\delta^{(3)}(\vec{x} - \vec{y})$$
$$[a(\vec{k}), a^\dagger(\vec{k}')] = [b(\vec{k}), b^\dagger(\vec{k}')] = (2\pi)^3 2\omega\delta^{(3)}(\vec{k} - \vec{k}')$$
where $\Pi = \partial_0\phi^\dagger$ and $\Pi^\dagger = \partial_0\phi$.
- The fourier decomposition of a function $f(x)$ is defined as
$$f(x) := \int_{-\infty}^{\infty} \frac{d^4k}{(2\pi)^4} \hat{f}(k)\, e^{ikx}$$
where $\hat{f}$ is the fourier transform of f.

Exercise 1: Short questions [20 pts + 2 bonus]

Answer the following questions, try to be as brief as possible.

(i) **[4 pts]** State Wick's theorem for a free real scalar field.

(ii) **[4 pts]** When calculating correlators using the generating function expressed as a path integral, why does one need to use the time-ordering operator?

(iii) **[3 pts]** What are the two types of transformations found in the proper orthochronous part of the Lorentz group?

(iv) **[4 pts]** Write down the expression for a free real scalar field in terms of the creation and annihilation operators.

(v) **[5 pts + 2 bonus]** Explain why it is necessary to introduce an $i\epsilon$ in the path integral, by deforming the integration contour (or by adding it in the Lagrangian). **Bonus:** Give a physical interpretation of this.

Exercise 2: Complex scalar field propagator [30 pts + 5 bonus]

The propagator of a free complex scalar field is obtained by the following time-ordered product

$$i\langle 0|T\phi(x)\phi^\dagger(y)|0\rangle \qquad (1)$$

In this exercise we will derive a concrete expression for this time-ordered product.

(i) **[5 pts]** First show that we can write

$$(-\partial_x^2 + m^2)\Big(\theta(x^0 - y^0)\langle 0|\phi(x)\phi^\dagger(y)|0\rangle\Big) =$$

$$= \partial_{x^0}\Big(\delta(x^0 - y^0)\langle 0|\phi(x)\phi^\dagger(y)|0\rangle\Big) + \delta(x^0 - y^0)\partial_{x^0}\langle 0|\phi(x)\phi^\dagger(y)|0\rangle$$

Hint: use that $\partial_x\theta(x) = \delta(x)$.

(ii) **[10 pts]** Using this, check that (1) is the Green's function of the Klein-Gordon operator.

(iii) **[10 pts]** Expressing the complex quantum fields $\phi(x)$, $\phi^\dagger(y)$ in terms of the annihilation and creation operators $a(k)$, $a^\dagger(k)$, $b(k)$ and $b^\dagger(k)$, rewrite (1) as:

$$\Delta(x-y) = i\theta(x^0 - y^0)\int \mathrm{d}\tilde{k}\ e^{ik\cdot(x-y)} + i\theta(y^0 - x^0)\int \mathrm{d}\tilde{k}\ e^{-ik\cdot(x-y)} \qquad (2)$$

where $\mathrm{d}\tilde{k} = \frac{\mathrm{d}^3\vec{k}}{2\omega(\vec{k})(2\pi)^3}$ and $\omega = \sqrt{\vec{k}^2 + m^2}$.

(iv) **[5 pts]** Argue whether or not (2) is invariant under the orthochronous subgroup of Lorentz transformations.

(v) **[Bonus, 5 pts]** Finally, verify that (1) corresponds to the Feynman propagator you derived in class using path integral quantization, i.e. show that (2) can be rewritten as:

$$\Delta(x-y) = \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{e^{ik\cdot(x-y)}}{k^2+m^2-i\epsilon} \tag{3}$$

with $\epsilon > 0$ infinitesimal.

Exercise 3: Complex scalar field generating function [25 pts + 3 bonus]

Consider a theory with two free real scalar fields with equal mass

$$\mathcal{L}_0 = -\frac{1}{2}\partial_\mu\varphi_1\partial^\mu\varphi_1 - \frac{1}{2}m^2\varphi_1^2 - \frac{1}{2}\partial_\mu\varphi_2\partial^\mu\varphi_2 - \frac{1}{2}m^2\varphi_2^2 . \tag{4}$$

(i) **[3 pts]** This theory is equivalent to a complex scalar field theory, by defining a complex scalar $\phi = \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2)$. Write down the Lagrangian density for ϕ.

(ii) **[10 pts]** Write down the path integral generating function $Z_0[J^\dagger, J]$ for the complex scalar Lagrangian density you get in (i), with source terms $J^\dagger\phi + \phi^\dagger J$. Then use Fourier transforms to express $Z_0[J^\dagger, J]$ in momentum space.

(iii) **[7 pts]** Calculate the generating function $Z_0[J^\dagger, J]$ in momentum space via the Gaussian integral. You do not have to determine the overall constant. Hint: Your result should include the Feynman propagator with expression (3) (the $i\epsilon$ term is not important here).

(iv) **[5 pts]** Calculate the 4-point function $\langle 0| T\phi^\dagger(x_1)\phi^\dagger(x_2)\phi(x_3)\phi(x_4) |0\rangle$.

(v) **[Bonus, 3 pts]** Now add an interaction term $\mathcal{L}_{\text{int}} = -\frac{\lambda}{4}\left(\phi^\dagger\phi\right)^2$ to the Lagrangian density of the complex scalar field. Express the new generating function $Z[J^\dagger, J]$, which possesses this interaction term, in terms of the functional derivatives of $Z_0[J^\dagger, J]$.

Exercise 4: Spontaneous Symmetry Breaking [25 pts]

Consider two interacting real scalar fields, ϕ_1, ϕ_2, described by the Lagrangian density

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 - \frac{1}{2}m_0^2\phi_1^2 - \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - \frac{1}{2}m_0^2\phi_2^2 - \frac{\lambda}{4}\left(\phi_1^2+\phi_2^2\right)^2 , \tag{5}$$

where $\lambda > 0$.

(i) **[5 pts]** Show that $\mathcal{L}$ is invariant under the transformation

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \to R(\theta)\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \tag{6}$$

where θ is a real parameter independent of spacetime. This invariance is called *global $O(2)$ symmetry.*

Now define the scalar potential,

$$V[\phi_1, \phi_2] := \frac{1}{2}m_0^2\phi_1^2 + \frac{1}{2}m_0^2\phi_2^2 + \frac{\lambda}{4}\left(\phi_1^2 + \phi_2^2\right)^2, \tag{7}$$

such that the Lagrangian density reads,

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 - \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - V[\phi_1, \phi_2]. \tag{8}$$

(ii) **[5 pts]** Assume $m_0^2 > 0$, sketch the potential $V[\phi_1, \phi_2]$, and show that it has a unique minimum.

The minimum that you have found is the unique physical vacuum, which also satisfies $\langle\phi_1\rangle = \langle\phi_1\rangle = 0$, i.e., vanishing vacuum expectation value (VEV).

For the remaining of this question, we assume $\lambda \ll 1$ such that the perturbative approximation gives a good enough description of the theory. Within this approximation, the squared mass is the eigenvalues of the matrix $\frac{\partial^2 V}{\partial\phi_i\partial\phi_j}$ around the minimum, where $i, j \in \{1, 2\}$.

(iii) **[4 pts]** Compute the mass(es) of the fields that are given by the above explanation. Are they finite or do they vanish?

(iv) **[5 pts]** Assume $m_0^2 < 0$, and show that the minima of $V[\phi_1, \phi_2]$ satisfies $\phi_1^2 + \phi_2^2 = v^2$ where v is a constant that you should determine.

Notice that the potential has a continuum of minima for this case. This means the system has infinitely many candidates for the vacuum state. However, the system has to *choose* a vacuum since it is a unique state. This choice leads to the concept of *spontaneously broken global symmetry.*

Now, without loss of generality, choose the vacuum state to be at $\bar{\phi}_1 = v$, $\bar{\phi}_2 = 0$ and define the fields $\sigma(x) := \phi_1(x) - v$ and $\pi(x) := \phi_2(x)$. This gives you a new Lagrangian in terms of σ and π. Note that the new Lagrangian does not have global $O(2)$ symmetry anymore, therefore making this an example of spontaneous symmetry breaking.

(v) **[6 pts]** Write down the Lagrangian density in terms of σ, π, λ, v. Calculate the masses of the fields σ, π.