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In 2005/2006, the course NS-TP401M was given by prof. dr. B.Q.P.J. de Wit.

Quantum Field Theory (retake exam) (NS-TP401M)
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Question 1. Pion annihilation into charged kaons

We consider the annihilation of charged pions into charged kaons, mediated through the exchange of a virtual photon.

a) We start by considering the Lagrangian for a complex scalar field ϕ interacting with a photon field A_μ,

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)^2 - \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi \\ &- \mathrm{i}eA_\mu\left[\phi^*(\partial_\mu\phi) - (\partial_\mu\phi^*)\phi\right] - e^2A_\mu^2\phi^*\phi.\end{aligned} \tag{1}$$

This Lagrangian will be used to describe the interaction of electrically charged pions $\pi^\pm$ and of electrically charged kaons $K^\pm$ with photons. Pions and kaons are spinless elementary particles with masses of about 140 and 494 MeV/c^2 and lifetimes of the order of 10^{-8} seconds. Hence the scalar field ϕ can be associated with either $\pi^\pm$ or $K^\pm$ particles with corresponding masses m_π and m_K. Demonstrate that the Lagrangian (1) is invariant under the combined gauge transformations.

$$A_\mu(x) \to A_\mu(x) + \partial_\mu\xi(x), \quad \phi(x) \to e^{\mathrm{i}e\xi(x)}\phi(x). \tag{2}$$

b) We consider the process fo pion annihilation, in which charged pions annihilate into a virtual photon which subsequently decays into two charged kaons: $\pi^+ + \pi^- \to K^+ + K^-$. We describe this process in tree approximation by the exchange of a virtual photon. Write down the interaction vertices of the photon with an incoming $\pi^\pm$ pair with four-momenta p_+ and p_-, and of the photon with an outgoing $K^\pm$ pair with momenta k_+ and k_-. Give the constraints on these momenta when the pions and kaons are on their respective mass shells. Draw the relevant Feynmann diagram for this process indicating all the momenta of the external and internal lines. Indicate the direction of the charge by arrows on the lines.

c) Consider the definition of the photon propagator and argue that it does not exist on the basis of the Lagrangian (1). To cure this problem introduce a so-called gauge-fixing term in the Lagrangian,

$$\mathcal{L}_{\text{gauge-fixing}} = -\frac{1}{2}\left(\lambda\partial^\mu A_\mu\right)^2, \tag{3}$$

with λ an arbitrary parameter. Calculate now the photon propagator.

d) Use the previous results to write down the amplitude for the process $\pi^+ + \pi^- \to K^+ + K^-$, with the kaon and pion momenta on their respective mass shells. Show that the result is independent of λ. Do you understand this independence?

e) Show that the invariant amplitude can be written in simple form,

$$\mathcal{M} = e^2\frac{u-t}{s}, \tag{4}$$

where s, t and u are the so-called Mandelstam variables,

$$s = -(p_+ + p_-)^2, \quad t = -(p_+ - k_+)^2, \quad u = -(p_+ - k_-)^2.$$

Comment: This particular process is itself not very relevant experimentally. But it is a good prototype for understanding the similar process of electron-positron annihilation into heavy lepton or quark pairs. This process is very important experimentally.

Question 2. Renormalizability of massive vector fields

Consider a vector field A_μ coupled to a real scalar field ϕ and spinor field ψ in four space-time dimensions, described by the Lagrangian

$$\mathcal{L} = -\frac{1}{4}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)^2 - \frac{M^2}{2q^2}\left|(\partial_\mu - \mathrm{i}qA_\mu)e^{\mathrm{i}q\phi/M}\right|^2 - \bar{\psi}\left(\partial\!\!\!/ - \mathrm{i}gA\!\!\!/ + m\right)\psi.$$

a) Show that the Lagrangian is invariant under the combined gauge transformations

$$\begin{aligned} A_\mu &\rightarrow A_\mu + \partial_\mu \xi, \\ \phi &\rightarrow \phi + M\,\xi, \\ \psi &\rightarrow \exp\left[\mathrm{i}g\xi\right]\psi, \end{aligned}$$

where $\xi(x)$ is an arbitrary function of space and time.

b) Collect all terms quadratic in the fields A_μ and ϕ. Argue that the inverse propagator takes the form of a 5×5 matrix and determine the matrix. Does the propagator exist? (Try to motivate the answer in two different ways: both on the basis of the explicit matrix and on the basis of a more general argument.)

c) Argue that $\phi = 0$ is an admissable gauge condition. Determine now the propagator for A_μ in this gauge. What are the physical bosonic states of given momentum described by the resulting Lagrangian? (Note: we do not ask for a detailed derivation.)

d) Is the theory renormalizable by power counting and why (not)? Write down the expression for the fermion self-energy diagram in the one-loop approximation (there is no contribution from tadpole diagrams, so one has only one diagram to consider) and determine the degree of divergence of the corresponding integral. What kind of counterterms do you expect to need in order to absorb the infinities of the integral? (Give qualitative arguments; do *not* calculate the integral or the coefficients of these counterterms.)

e) We now choose another gauge condition by adding the following term to the Lagrangian,

$$\mathcal{L}_{\mathrm{gauge-fixing}} = -\frac{1}{2}(\lambda\partial_\mu A^\mu - M\lambda^{-1}\phi)^2,$$

with λ an arbitrary parameter. Calculate again the propagators for A_μ and ϕ. What are in this case the physical bosonic states for given momentum described by the correspondenting Lagrangian. Compare your result with your previous answer and give your comments.

f) In the last formulation, is the theory renormalizable by power counting and why (not)? Write down the expression for the fermion self-energy diagram in the one-loop approximation and determine again the degree of divergence in the integral. In this case, what are the counterterms that are needed in order to absorb the infinities?

g) **Bonus question:** Is the theory now fully renormalizable or not?

Question 3. Instantons and fermions

We consider a quantum mechanical model based on a bosonic coordinate x and a fermionic coordinate ψ. The latter is a real, two-component spinor. Formulated in Euclidean time τ, the action equals

$$S_E = \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}\tau \left\{ \dot{q}(\tau)^2 + U^2(q) + \psi^{\mathrm{T}}\dot{\psi} + (\psi^{\mathrm{T}}\sigma_2\psi)\frac{\partial U(q)}{\partial q} \right\},$$

where ψ^{T} is the transposed of ψ and σ_2 is the matrix $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

a) Derive the equations of motion for $q(\tau)$ and $\psi(\tau)$.

b) Consider the (infinitesimal) transformation rules

$$\delta q = \varepsilon^{\mathrm{T}} \sigma_2 \psi; \qquad \delta \psi = \dot{q} \sigma_2 \varepsilon - U(q) \varepsilon, \tag{5}$$

where ε is a time-independent, anticommuting two-component spinor which serves as a parameter of the transformation. Write down the variation $\delta \psi^{\mathrm{T}}$. Show that the Lagrangian transforms into a total derivative. [**Hints:** Realize that ψ is anti-commuting, this implies that $\varepsilon^{\mathrm{T}} \sigma_2 \psi = \psi^{\mathrm{T}} \sigma_2 \varepsilon$, and furthermore that any product of more than two identical spinors must vanish!] Hence the action is invariant under (5). Consider first the simple case when $U = 0$, and then prove invariance for general $U(q)$.

c) Consider the double-well potential, where

$$V(q) = \frac{1}{2} U^2(q), \qquad \text{with} \quad U(q) = \frac{1}{2} \sqrt{\lambda} \left(q^2 - \frac{\mu^2}{\lambda} \right).$$

Observe that $U \leq 0$ when q is located between the two minima of the potential. The usual instanton solution is obtained by setting the fermions to zero and choosing

$$q_0(\tau) = \frac{\mu}{\lambda} \tanh \left[\frac{1}{2} \mu (\tau - \tau_0) \right], \qquad \psi_0 = 0.$$

Here τ_0 is a collective coordinate associated with Euclidean time translations. The instanton solution satisfies $\dot{q}(\tau) + U(q(\tau)) = 0$ and has finite action. The anti-instanton satisfies $\dot{q}(\tau) - U(q(\tau)) = 0$. Show that the solutions of these linear differential equations satisfy the field equation which is a second-order differential equation.

d) There is another solution, for which the two-component spinor is non-zero. Such a solution follows from the observation that $q(\tau) = q_0(\tau) + \delta q(\tau)$ and $\psi(\tau) = \psi_0(\tau) + \delta \psi(\tau)$, where δq and $\delta \psi$ are the result of the transformation (5) on the previous solution, must satisfy the equations of motion. Show that this yields

$$q_0(\tau) = \frac{\mu}{\lambda} \tanh \left[\frac{\mu}{\sqrt{2}} (\tau - \tau_0) \right], \qquad \psi(\tau) = \xi f(\tau) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix},$$

where we have introduced a (constant) Grassman parameter ξ.

e) Explain why ξ is called a fermionic collective coordinate.

f) Give also the solution with non-zero ψ for the anti-instanton.

g) **Bonus question:** Consider performing a semi-classical evaluation where one integrates over small bosonic and fermionic fluctuations around the instanton solution. Can you understand why the result will vanish?