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In 2004/2005, the course NS-TP401m was given by prof. dr. B.Q.P.J. de Wit.

Quantum Field Theory (NS-TP401m)
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Question 1: Scalar field on a circle

Consider the action of a real scalar field in two spacetime dimensions,

$$S = \int \mathrm{d}x\, \mathrm{d}t\, \left(-\tfrac{1}{2}(\partial_\mu\phi)^2 - \tfrac{1}{2}m^2\phi^2 - \lambda\phi^4\right). \tag{1}$$

a) Assume that the spatial component x parametrizes a circle of length $L = 2\pi R$, and decompose the scalar field in terms of a Fourier sum. Indicate the possible values of the momenta.

b) Express the action in terms of these Fourier modes and show that you obtain a quantum-mechanical model of an infinite tower of harmonic oscillators $\phi_n(t)$, where $n = 0, \pm 1, \pm 2, \ldots$, with frequencies (masses),

$$M_n^2 = m^2 + \frac{n^2}{R^2}. \tag{2}$$

Normalize the ϕ_n such that kinetic energy reads $\frac{1}{2}(\partial_t\phi_0)^2 + \sum_{n>0} |\partial_t\phi_n|^2$.

c) Write down the propagators and vertices in momentum space for this quantum-mechanical model.

d) Draw the Feynman diagram(s) that contribute to the self-energy of ϕ_0 in the one-loop approximation. In the same approximation, compute the full propagator and the correction to the ϕ_0-mass. Use the fact that the propagator,

$$\Delta(x-y) = \frac{1}{i(2\pi)^d}\int \mathrm{d}^d k \frac{\mathrm{e}^{ik_\mu(x-y)^\mu}}{k^2 + M^2 - i\epsilon}, \tag{3}$$

for $d = 1$, is given by $\Delta(x-y) = \frac{1}{2M}\mathrm{e}^{-iM|x-y|}$.

e) Present the separate contributions to the ϕ_0-mass from the ϕ_0-propagator and from the $\phi_{\pm|n|\neq 0}$-propagators (for given $|n|$). Consider now the limits $R \to 0$ and $R \to \infty$, assuming that $\lambda' = \lambda/L$ is kept constant.

f) Add the various contributions and discuss the result for the ϕ_0-mass in the two limits.

Question 2: Coherent states quantization

Consider the path integral in the phase-space representation where the action equals $S[p(t), q(t)] = \int \mathrm{d}t[p\dot{q} - H(p,q)]$. Subsequently choose the complex variable

$$a = \frac{1}{\sqrt{2\omega}}(\omega q + ip). \tag{4}$$

We assume $\hbar = 1$.

a) Write the action in terms of $a(t)$ and $a^*(t)$ and show that, up to boundary terms it is equal to

$$S[a(t), a^*(t)] = \int \mathrm{d}t[ia^*\dot{a} - H(a^*, a)]. \tag{5}$$

Derive the equations of motion for a and a^* from Hamilton's principle. Do not worry about the precise boundary values for a and *.

b) Determine $H(a^*, a)$ for the harmonic oscillator (we choose $m = 1$ and select the same ω as above),

$$H = \tfrac{1}{2}(p^2 + \omega^2 q^2). \tag{6}$$

c) Note that this system has only first-order in time derivatives. Determine the momentum conjugate to a and derive the canonical commutation relations for the operators a and a^*.

d) View this system as a 'field' theory based on two fields, a and a^*. What is the propagator when using the Hamiltonian $H(a^*, a)$ that you derived in question b)?.

e) Show that a quantum-mechanical representation for the operators a and a^* is given in the "z-representation" by

$$a^* = z, \qquad a = \frac{\mathrm{d}}{\mathrm{d}z}. \tag{7}$$

In this representation a and a^* act on wavefunctions $\psi(z)$, where z is complex. Hence the wavefunctions are holomorphic.

f) Show that the functions $\psi_\lambda(z) \propto \exp(\lambda z)$ are eigenfunctions of the operator a with eigenvalue λ. These are the so-called coherent states. Show that the monomials $\psi_n(z) \propto z^n$ are eigenfunctions of the occupation number operator $a^* a$.
(**Remark**: In this representation hermitean conjugation and the normalizability of wavefunctions is not so obvious. You may ignore these aspects here.)

Question 3: An auxiliary field

Consider a field theory (in four space-time dimensions) with two real fields, ϕ and A, described by the Lagrangian

$$\mathcal{L} = -\tfrac{1}{2}(\partial_\mu \phi)^2 - \tfrac{1}{2}m^2\phi^2 - \lambda\phi^4 - \tfrac{1}{2}A^2 + A(\mu^2 + g\phi^2). \tag{8}$$

a) Give the propagators and vertices.

b) Calculate the self-energy diagrams in the tree approximation and give the masses for the *physical* particles described in this approximation. Give the full propagators in the tree approximation and use these in the next three questions.

c) Calculate the (three) self-energy diagrams for the field ϕ in the one-loop approximation. Give the mass-shift of ϕ in that approximation. (Note: do not try to evaluate the integrals.) For which value of g does the mass shift vanish?

d) Solve the (classical) equations of motion for A and substitute the result into the Lagrangian, which will then depend only on ϕ. Show that this corresponds to integrating out the field A in the path integral.

e) Evaluate now again the mass of the field ϕ in tree approximation and compare the result to the answers to question b) above.

f) Calculate again the self-energy diagrams in the one-loop approximation. Compare the result to the results obtained in question c). Can you now explain the cancellation noted in question c), noted for special values of g?

g) **Bonus question** Calculate the mass shift for the field A in the one-loop approximation. What do you conclude?