Instituut voor Theoretische Fysica
Universiteit Utrecht

MIDTERM EXAM Quantum Field Theory

Thursday, November 15 2007, 14.00 - 17.00, Rooms BBL 105B, Minnaert 205.

1) Start every exercise on a separate sheet.

2) Write on each sheet: your name and initials. In addition, write on the *first* sheet: your address, postal code and your field of study. Indicate whether you follow the master's programme in theoretical physics.

3) Please write legibly and clearly.

4) The exam consists of **three** exercises.

1. Functions versus functionals

Consider a functional

$$F[f] = \int \mathrm{d}x \, f(x) \,. \tag{1}$$

i) Interpret this expression by discretizing the continuous variable x into N independent variables x_i separated by an increment Δ. The continuum limit amounts to shrinking $\Delta \to 0$ keeping the range of integration $N\Delta$ fixed. Write down (1) as a sum over the function values $f_i = f(x_i)$.

ii) Define the functional derivative $\partial F[f]/\partial f(x)$ of (1) by considering infinitesimal changes of the function $f(x)$. Rewrite this definition in the limiting procedure.

iii) In the discrete case one would expect that the functional derivative corresponds to the partial derivative with respect to the variables f_i. Can you verify whether or not this is the case? If not, can you write down the correct expression for the functional derivative in the dicretized case, such that you obtain the same answer as for the functional derivative?

iv) Now consider some general functional $F[f]$ subject to the functional differential equation,

$$\frac{\partial F[f]}{\partial f(x)} = \lambda(x) \, F[f] \,. \tag{2}$$

Derive the solution by first solving the discretized equation and subsequently taking the continuum limit.

2. Tunneling for a particle on a circle

Consider a particle with mass m moving on a circle with radius R, described by an angular variable $0 \le \theta < 2\pi$. The Euclidean Lagrangian is given by,

$$L^E = \tfrac{1}{2}mR^2 \, \dot{\theta}^2 + V(\theta) \,, \tag{3}$$

with potential $V(\theta) = g^2 \sin^2(N\theta/2)$. This potential is non-negative and has N minima at regular distances around the circle. Clearly there exists a (anti-)instanton solution connecting two adjacent minima, which we denote by $\theta_0(\tau)$.

i) Prove that the potential has minima at $\theta = 2\pi n/N$ where $0 \le n < N - 1$, and that $\mathrm{d}^2V/\mathrm{d}\theta^2$ equals $\frac{1}{2}N^2g^2$ at these minima. It is convenient to inroduce the quantity $\omega = gN/\sqrt{m\,R^2}$. What is the meaning of ω? (g will be taken positive).

ii) Prove that the instanton solution satisfies the differential equation,

$$\frac{\mathrm{d}\theta(\tau)}{\mathrm{d}\tau} = \frac{\omega\,\sqrt{2}}{N}\,\sin(N\theta/2)\,. \tag{4}$$

and that the corresponding action equals ($N \ne 0$),

$$S_0 = S^E[\theta_0] = g\,\sqrt{2mR^2}\int_0^{2\pi/N} \mathrm{d}\theta\,\sin(N\theta/2) = \frac{4\sqrt{2}\,g^2}{\omega}\,. \tag{5}$$

iii) Let us now consider the Euclidean path integral in the semiclassical approximation in the limit that $\tau_2 - \tau_1 = T \to \infty$ between two minima $\theta_1 = 2\pi m/N$ and $\theta_2 = 2\pi n/N$, where m, n are integers between 0 and $N - 1$. Discuss all the classical paths that contribute to the functional integral in the infinite-T limit. Note that there is an extra degeneracy as compared to the case of the periodic potential, due to the fact that the particle moves on a circle so that the minima of the potential are identified modulo N.

iv) In the lectures we derived the expression for the Euclidean path integral between two minima at ma and na in a periodic potential on an infinite line, where a is the periodicity interval (which we identify with $a = 2\pi/N$) and m and n are integers,

$$W^{\mathrm{E}}(ma, \infty\,;\, na, -\infty) =$$
$$\lim_{T \to \infty} \sqrt{\frac{mR^2\omega}{\pi\hbar}}\,\mathrm{e}^{-\omega T/2} \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta}{2\pi}\,\mathrm{e}^{\mathrm{i}(n-m)\theta}\,\exp\left[2\cos\theta\sqrt{\frac{S_0}{2\pi\hbar}}K'\,T\,\mathrm{e}^{-S_0/\hbar}\right],\tag{6}$$

Argue that the corresponding expression on the circle takes the form

$$W^{\mathrm{E}}(2\pi n/N, \infty\,;\, 2\pi m/N, -\infty) =$$
$$\sum_{k=-\infty}^{\infty} W^{\mathrm{E}}(2\pi n/N + 2\pi k, \infty\,;\, 2\pi m/N, -\infty)\,,\tag{7}$$

where $0 \le m, n < N - 1$.

v) Using the Poisson resummation formula

$$\sum_{k=-\infty}^{\infty} \mathrm{e}^{2\pi \mathrm{i}kx} = \sum_{l=-\infty}^{\infty} \delta(x - l)\,,\tag{8}$$

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combine (6) and (7) to derive the following result,

$$W^{\mathrm{E}}(2\pi n/N, \infty\,;\, 2\pi m/N, -\infty) =$$

$$\lim_{T\to\infty} \frac{1}{N} \sqrt{\frac{mR^2\omega}{\pi\hbar}} \sum_{l=-[N/2]}^{[N/2]} \mathrm{e}^{2\pi \mathrm{i} l(n-m)/N}\, \mathrm{e}^{-E_l\, T/\hbar} \,, \tag{9}$$

where $[N/2]$ is the integer part of $N/2$ (i.e. $[N/2]$ equals $N/2$ for even N and $(N-1)/2$ for odd N), and

$$E_l = \tfrac{1}{2}\hbar\omega - 2\hbar \cos\frac{2\pi l}{N} \sqrt{\frac{S_0}{2\pi\hbar}} K'\, \mathrm{e}^{-S_0/\hbar} \,. \tag{10}$$

vi) **Bonus question** Argue that the above result comprises the contributions of an odd number of states. One is the ground state, which is non-degenerate. The excited energy levels are all degenerate. Can you explain the presence of the factor $\exp[2\pi \mathrm{i} l(n-m)/N]$?

3. An effective action

We consider two scalar fields $\phi(x)$ and $\eta(x)$ defined in a $(3+1)$-dimensional spacetime and the following two Lagrangians,

$$\begin{aligned}
\mathcal{L}_A &= -\tfrac{1}{2}(\partial_\mu\phi(x))^2 - \tfrac{1}{2}m^2\phi^2(x) - \lambda\,\phi^4(x)\,,\\
\mathcal{L}_B &= -\tfrac{1}{2}(\partial_\mu\phi(x))^2 - \tfrac{1}{2}m^2\phi^2(x) - g\,\eta(x)\,\phi^2(x) - \tfrac{1}{2}\mu^2\eta^2(x)\,.
\end{aligned} \tag{11}$$

These Lagrangians specify quantum field theories that we will call theory A and theory B, respectively.

i) Determine the Feynman rules for both theories.

ii) Draw, in both theories, the tree diagrams with four external ϕ-lines and derive a relationship between the coupling constant λ from theory A and and the coupling constants g and μ from theory B such that they produce identical results.

iii) Give arguments based on the field equations that the two Lagrangians (11) describe the same quantum field theory.

iv) Give arguments based on the path integral that support your conclusion regarding the previous question.

v) Give the physical masses associated with the fields ϕ and η in tree approximation.

vi) Consider the path integral for theory B and perform the (Gaussian) integration over $\phi(x)$ so that we are left with a result of the form

$$\int \mathcal{D}\eta\, \exp\Big(-\frac{i}{\hbar}S_{\mathrm{eff}}[\eta(x)]\Big)\,. \tag{12}$$

Give a formal expression for $S_{\mathrm{eff}}[\eta(x)]$. *Hint*: use that $\mathrm{Det}(A) = \exp\left(\mathrm{Tr}[\log(A)]\right)$. Drop the contributions of zeroth order in g and write the propagator as a bilocal quantity $\Delta(x,y)$.

vii) Give the order g and g^2 terms in the expansion of $S_{\mathrm{eff}}[\eta(x)]$. Identify the corresponding Feynman diagrams from which these terms originate.

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do not consider this This is one option:

1. Consider a theory of two real scalars ϕ, A in four dimensions interacting with a three vertex, given by the Lagrangian:

$$L = \tfrac{1}{2}(\partial_\mu \phi)^2 - \tfrac{1}{2}(\partial_\mu A)^2 - \tfrac{1}{2}m^2 A^2 + gA\phi^2 \tag{13}$$

Give the propagators and vertices for this theory, including the $-i\varepsilon$ in the expression of the propagators. The reason will be seen later.

2. Compute the amplitude for the three diagrams with four external ϕ particles at tree level. From now on we will concentrate on the diagram in which the two incoming ϕ's are annihilated into an A.

3. Give the rough behaviour of this diagram as a function of the total incoming momentum p_{tot} when the limit $\varepsilon \to 0$ is taken. Here you may go to the center of mass frame in which the sum of the incoming spatial momenta is zero, so that the amplitude only depends on the total incoming energy E_{tot}.

Can the result be physically acceptable for energies near the mass of the A particle, given that the probability for any process to happen is proportional to the absolute square of the amplitude?

4. In order to clarify this, we consider the one loop correction to the propagator of A. Give the expression corresponding to the selfenergy of A to that order. Do not compute the integral and denote this expression as $\Sigma(p)$. Note that $\Sigma(p)$ is in general a complex function of p.

5. Including this correction to the propagator means that the selfenergy diagram must be added to the expression for the propagator. Argue that the new propagator is (**add figure??**):

$$\Delta_D = \frac{1}{(2\pi)^4 i}\frac{1}{p^2 + m^2 - i\varepsilon} + \frac{1}{(2\pi)^4 i}\frac{1}{p^2 + m^2 - i\varepsilon}\Sigma(p)\frac{1}{(2\pi)^4 i}\frac{1}{p^2 + m^2 - i\varepsilon}. \tag{14}$$

To first order in the interaction, this is

$$\Delta_D = \frac{1}{i(2\pi)^4}\frac{1}{p^2 + m^2 - i\varepsilon - \frac{1}{i(2\pi)^4}\Sigma(p)}. \tag{15}$$

Use this 'dressed' propagator in the diagram considered in point 1. Take the absolute square of the amplitude to show that there is now no real value of the incoming momenta for which the probability of this process diverges. For this, note that the real part of $\Sigma(E)$ adds to ε, shifting the propagator pole away from m^2 even when the limit $\varepsilon \to 0$ is taken.

Bonus Question Can you say something about how the propagator would look like if not one, but an infinite number of such selfenergy loops were taken into account (**add figure??**)?

6. The general form of the amplitude squared now resembles a Gaussian. This is called a resonance in the particle physics literature. Can you think of a way to discover new particles with a very large mass experimentally using the above considerations?

not to be included:

There will be an extra sum over the number of times the instantons go around the circle, so we get:

$$
W_C(\frac{2\pi m}{N}, \infty \,; \frac{2\pi n}{N}, -\infty) = \sum_{k=-\infty}^{\infty} W(\frac{2\pi m}{N} + 2\pi k, \infty \,; \frac{2\pi n}{N}, -\infty)
$$

$$
= \lim_{T\to\infty} \frac{gN}{\sqrt{2\pi\hbar}} e^{-g^2 N^2 T/4} \sum_{k=-\infty}^{\infty} \int \frac{d\theta}{2\pi} e^{i\theta(m-n+kN)} \exp[2\cos\theta \sqrt{\frac{S_0}{2\pi\hbar}} KTe^{-S_0/\hbar}]
$$

$$
= \lim_{T\to\infty} \frac{gN}{\sqrt{2\pi\hbar}} e^{-g^2 N^2 T/4} \sum_{l=-\infty}^{\infty} \int \frac{d\theta}{2\pi} e^{i\theta(m-n)} \delta(\frac{\theta N}{2\pi} - l) \exp[2\cos\theta \sqrt{\frac{S_0}{2\pi\hbar}} KTe^{-S_0/\hbar}]
$$

$$
= \lim_{T\to\infty} \frac{g}{\sqrt{2\pi\hbar}} e^{-g^2 N^2 T/4} \sum_{l=-\infty}^{\infty} e^{\frac{2\pi i l}{N}(m-n)} \exp[2\cos\frac{2\pi l}{N} \sqrt{\frac{S_0}{2\pi\hbar}} KTe^{-S_0/\hbar}] \tag{16}
$$

5. Compare your answer with the one for the infinite line case with a periodic potential. Can you explain the observed difference based on your knowledge of the cases of a free particle on a line and on a circle?

6. Recalling that the above amplitude is in fact the inner product $< \frac{2\pi m}{N}|e^{-HT/\hbar}|\frac{2\pi n}{N} >$, insert in this matrix element a set of eigenstates $|k>$, labelled by $k = 0, \ldots, N-1$. This is not a complete set, but it is sufficient in this approximation. Using the above result, show that these states must satisfy:

$$
<m|k><k|n> = \frac{g}{\sqrt{2\pi\hbar}} e^{\frac{2\pi i k}{N}(m-n)}, \qquad H|k> = \left(\frac{\hbar g^2 N^2}{4} - 2\hbar\cos\frac{2\pi k}{N} \sqrt{\frac{S_0}{2\pi\hbar}} Ke^{-S_0/\hbar} \right) |k> \tag{17}
$$

Argue that the first of these equations implies that the wavefunctions have the symmetry:

$$
<n+p|k> = e^{\frac{2\pi i k}{N}p} <n|k> \tag{18}
$$