Institute for Theoretical Physics
Utrecht University

Retake Exam Statistical Field Theory (NS-TP402M)

Tuesday, March 16, 2010, 14:00-17:00

1. *Use a separate sheet for every excercise.*
2. *Write your name and initials on all sheets, on the first sheet also your address and your student ID number.*
3. *Write clearly, unreadable work cannot be corrected.*
4. *You may use your notes and the book by Stoof et al.*
5. *Distribute your time evenly over all exercises, don't spend an enormous amount of time on correcting minus signs, factors of two and/or π, etc. In case you suspect that you have made a calculational error but don't have enough time to correct it, point it out in words.*

Exercise 1: Spin density wave in a one-dimensional Fermi gas (*70 points*)

In this exercise, we consider a one-dimensional homogeneous Fermi gas of spin one-half fermions with mass m. We model the interaction between the particles with a contact interaction $V(x-x') = V_0\delta(x-x')$ with $V_0 > 0$. Upon lowering the temperature, this system can undergo a phase transition to a so-called spin density wave, with a spin density $m_z(x,\tau)$ given by

$$\langle m_z(x,\tau)\rangle \equiv \sum_{\sigma\sigma'} \langle \phi^*_\sigma(x,\tau)\tau^z_{\sigma\sigma'}\phi_{\sigma'}(x,\tau)\rangle/2 = A\cos(qx+\theta)\ .$$

Here, τ^z is the Pauli matrix

$$\tau^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and A, q, and θ are real numbers, with A the amplitude of the spin density wave, θ the phase, and q its wave number.

To describe the phase transition, we start out with the grand-canonical partition function given by

$$Z = \int d[\phi^*_\uparrow]\, d[\phi_\uparrow]\, d[\phi^*_\downarrow]\, d[\phi_\downarrow] \exp\left\{-S[\phi^*_\uparrow,\phi_\uparrow,\phi^*_\downarrow,\phi_\downarrow]/\hbar\right\}\ , \tag{1}$$

where the action is given by

$$\begin{aligned} S[\phi^*_\uparrow,\phi_\uparrow,\phi^*_\downarrow,\phi_\downarrow] &= \int_0^{\hbar\beta} d\tau \int dx \phi^*_\uparrow(x,\tau)\left[\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - \mu\right]\phi_\uparrow(x,\tau) \\ &+ \int_0^{\hbar\beta} d\tau \int dx \phi^*_\downarrow(x,\tau)\left[\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - \mu\right]\phi_\downarrow(x,\tau) \\ &+ V_0 \int_0^{\hbar\beta} d\tau \int dx \phi^*_\uparrow(x,\tau)\phi^*_\downarrow(x,\tau)\phi_\downarrow(x,\tau)\phi_\uparrow(x,\tau)\ . \end{aligned} \tag{2}$$

a) (*5 points*) Show that the interaction can be split in two parts according to

$$V_0\phi^*_\uparrow(x,\tau)\phi^*_\downarrow(x,\tau)\phi_\downarrow(x,\tau)\phi_\uparrow(x,\tau) = \frac{V_0}{4}\left[\sum_\sigma \phi^*_\sigma(x,\tau)\phi_\sigma(x,\tau)\right]^2 - \frac{V_0}{4}\left[\sum_{\sigma\sigma'}\phi^*_\sigma(x,\tau)\tau^z_{\sigma\sigma'}\phi_{\sigma'}(x,\tau)\right]^2 .$$

In the remainder of this exercise we ignore the first term on the right-hand side of the above decomposition of the interaction.

b) *(10 points)* Decouple the remaining part of the interaction with a Hubbard-Stratonovich transformation to the spin density $m_z(x,\tau)$ and show that after the transformation we have that

$$Z = \int d[m_z] d\left[\phi_\uparrow^*\right] d\left[\phi_\uparrow\right] d\left[\phi_\downarrow^*\right] d\left[\phi_\downarrow\right] \exp\left\{-S\left[\phi_\uparrow^*, \phi_\uparrow, \phi_\downarrow^*, \phi_\downarrow, m_z\right]/\hbar\right\} .$$

with the action

$$\begin{aligned} S\left[\phi_\uparrow^*, \phi_\uparrow, \phi_\downarrow^*, \phi_\downarrow, m_z\right] &= \int_0^{\hbar\beta} d\tau \int dx \phi_\uparrow^*(x,\tau) \left[\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - \mu\right] \phi_\uparrow(x,\tau) \\ &+ \int_0^{\hbar\beta} d\tau \int dx \phi_\downarrow^*(x,\tau) \left[\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - \mu\right] \phi_\downarrow(x,\tau) \\ &+ \int_0^{\hbar\beta} d\tau \int dx \left\{V_0 m_z^2(x,\tau) - V_0 m_z(x,\tau)\left[\phi_\uparrow^*(x,\tau)\phi_\uparrow(x,\tau) - \phi_\downarrow^*(x,\tau)\phi_\downarrow(x,\tau)\right]\right\} . \end{aligned} \quad (3)$$

c) *(10 points)* Perform the functional integration over the fermion fields $\phi_\uparrow^*, \phi_\uparrow, \phi_\downarrow^*, \phi_\downarrow$, so that the expression for Z is written as

$$Z = \int d[m_z] \exp\left\{-S_{\text{eff}}[m_z]/\hbar\right\} ,$$

and give the exact but formal expression for the effective action $S_{\text{eff}}[m_z]$.

d) *(10 points)* Calculate the effective action up to second order in the field $m_z(x,\tau)$ and show that (up to irrelevant constants)

$$\begin{aligned} S_{\text{eff}}[m_z] &= \int_0^{\hbar\beta} d\tau \int dx \int_0^{\hbar\beta} d\tau' \int dx' m_z(x,\tau) \left[V_0\delta(x-x')\delta(\tau-\tau')\right. \\ &\left. + \frac{V_0^2}{\hbar} G_0(x-x';\tau-\tau') G_0(x'-x;\tau'-\tau)\right] m_z(x',\tau') + \mathcal{O}\left(m_z^4\right) . \end{aligned} \quad (4)$$

NB: You do not have to show that the terms beyond second order start at order m_z^4.

e) *(15 points)* To describe the phase transition we consider static, i.e., time-independent, configurations of the spin density, which we write as

$$m_z(x) = \sum_{\mathbf{k}} m_z(k) e^{ikx} .$$

Show that for these configurations the effective action becomes

$$S_{\text{eff}}[m_z] = V_0 \hbar\beta L \sum_k \left[1 + V_0 \Pi(k)\right] m_z^*(k) m_z(k) + \mathcal{O}\left(m_z^4\right) .$$

with

$$\Pi(k) = \frac{1}{L}\sum_q \frac{N_F(\epsilon_{k+q}) - N_F(\epsilon_q)}{\epsilon_{k+q} - \epsilon_q} ,$$

where L is the length of the system, $\epsilon_k = \hbar^2 k^2/(2m)$, and $N_F(x) = \left[e^{\beta(x-\mu)} + 1\right]^{-1}$ is the Fermi-Dirac distribution function with μ the chemical potential.

f) (*5 points*) Consider temperatures smaller than the Fermi temperature. Argue that $|\Pi(k)|$ is largest for $k = 2k_F$ with k_F the Fermi wave number. (Note that $\Pi(k)$ is negative.)

In view of this result, we keep only the $k = 2k_F$ component of the effective action, so that we take for the effective action

$$S_{\text{eff}}[m_z] = V_0\hbar\beta L\left[1 + V_0\Pi(2k_F)\right] m_z^*(2k_F)m_z(2k_F) + \mathcal{O}\left(m_z^4\right) .$$

g) (*5 points*) We define a Landau free energy density $f_L(|m_z(2k_F)|)$ according to $S_{\text{eff}}[m_z] = \hbar\beta L f_L(|m_z(2k_F)|)$. Argue that this Landau free energy describes the phase transition to the spin density wave, and give its wave vector q.

h) (*5 points*) Give the equation that determines the critical temperature of the transition without evaluating it.

i) (*5 points*) The Landau free energy is invariant for transformations according to $m_z(2k_F) \to m_z(2k_F)e^{i\Lambda}$, with Λ a real number. To which physical symmetry does this transformation correspond?

Exercise 2: Atom-molecule coupling (*30 points*)

Consider a gas of spin one-half fermionic atoms with mass m, in a box of volume V. A pair of these atoms can form a bound state with energy ϵ, i.e., a molecule. The action for the coupled atom-molecule system is given by

$$\begin{aligned} S\left[\phi_\uparrow^*, \phi_\uparrow, \phi_\downarrow^*, \phi_\downarrow \phi_{\rm m}^*, \phi_{\rm m}\right] &= \int_0^{\hbar\beta} d\tau \int d\mathbf{x} \sum_{\sigma\in\{\uparrow,\downarrow\}} \left\{\phi_\sigma^*(\mathbf{x},\tau)\left[\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2\nabla^2}{2m} - \mu\right]\phi_\sigma(\mathbf{x},\tau)\right\} \\ &+ \int_0^{\hbar\beta} d\tau \int d\mathbf{x} \left\{\phi_{\rm m}^*(\mathbf{x},\tau)\left[\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2\nabla^2}{4m} + \epsilon - 2\mu\right]\phi_{\rm m}(\mathbf{x},\tau)\right\} \\ &+ \int_0^{\hbar\beta} d\tau \int d\mathbf{x} \left[\lambda\phi_{\rm m}^*(\mathbf{x},\tau)\phi_\uparrow(\mathbf{x},\tau)\phi_\downarrow(\mathbf{x},\tau) + \lambda\phi_\downarrow^*(\mathbf{x},\tau)\phi_\uparrow^*(\mathbf{x},\tau)\phi_{\rm m}(\mathbf{x},\tau)\right], \end{aligned}$$

where μ is the chemical potential. The field $\phi_\sigma(\mathbf{x},\tau)$, with $\sigma \in \{\uparrow,\downarrow\}$ labeling the two spin states, describes the atoms, and the field $\phi_{\rm m}(\mathbf{x},\tau)$ describes the molecules. The molecule-molecule interaction and atom-atom interaction has been neglected.

a) (*5 points*) Give the physical meaning of the last two terms in the action.

b) (*10 points*) Derive an effective action for the atoms by performing the Gaussian integral over the molecule field $\phi_{\rm m}(\mathbf{x},\tau)$, and show that this leads to an interaction term for the atoms, equal to

$$\frac{\lambda^2}{\hbar}\int_0^{\hbar\beta} d\tau \int d\mathbf{x} \int_0^{\hbar\beta} d\tau' \int d\mathbf{x}'\ \phi_\uparrow^*(\mathbf{x},\tau)\phi_\downarrow^*(\mathbf{x},\tau)G_{\rm m}(\mathbf{x},\tau,\mathbf{x}',\tau')\phi_\downarrow(\mathbf{x}',\tau')\phi_\uparrow(\mathbf{x}',\tau'),$$

where $G_{\rm m}(\mathbf{x},\tau,\mathbf{x}',\tau')$ is the molecule propagator. Draw the Feynman diagram corresponding to this interaction.

c) (*5 points*) Give the expression for $G_{\rm m}(\mathbf{x},\tau,\mathbf{x}',\tau')$ as a Fourier expansion.

d) (*10 points*) Consider the approximation in which the propagator of the molecules is frequency independent, i.e., $G_{\rm m}(\mathbf{k}, i\omega_n) \approx G_{\rm m}(\mathbf{k}, -2\mu/\hbar)$. Show that this approximation leads to an interaction of the form

$$\int_0^{\hbar\beta} d\tau \int d\mathbf{x} \int d\mathbf{x}'\phi_\uparrow^*(\mathbf{x},\tau)\phi_\downarrow^*(\mathbf{x},\tau)V(\mathbf{x}-\mathbf{x}')\phi_\downarrow(\mathbf{x}',\tau)\phi_\uparrow(\mathbf{x}',\tau) .$$

and show that

$$V(\mathbf{x}-\mathbf{x}') \propto \frac{e^{-|\mathbf{x}-\mathbf{x}'|/\xi}}{|\mathbf{x}-\mathbf{x}'|} \ ,$$

and determine the length ξ, and in particular its dependence on ϵ.