INSTITUTE FOR THEORETICAL PHYSICS
UTRECHT UNIVERSITY

Final Exam Statistical Field Theory (NS-TP402M)

Tuesday, 2 February, 2010, 14:00-17:00

1. *Use a seperate sheet for every excercise.*
2. *Write your name and initials on all sheets, on the first sheet also your address and your student ID number.*
3. *Write clearly, unreadable work cannot be corrected.*
4. *You may use your notes and the book by Stoof et al.*

Pseudospin ferromagnetism (*70 points*)

In this exercise, we consider a double-layer system of two two-dimensional homogeneous electron gases with equal density n of electrons that are parallel to each other and only interact via interlayer Coulomb interactions. There is no tunneling of electrons between the layers. We neglect the intralayer interactions, i.e., the electron-electron interactions within the layers. Furthermore, we ignore the spin of the electrons. This system is often modeled as a pseudospin system, meaning that we assign a spin one-half quantum number to the "which layer" degree of freedom: spin "up" ($\uparrow$) means that an electron is in the top layer, spin "down" ($\downarrow$) means bottom layer. We model the interlayer interaction by a contact interaction $V(\mathbf{x}-\mathbf{x}') = V_0\delta(\mathbf{x}-\mathbf{x}')$ with $V_0 > 0$.

The starting point is the grand-canonical partition function given by

$$Z = \int d\left[\phi_\uparrow^*\right] d\left[\phi_\uparrow\right] d\left[\phi_\downarrow^*\right] d\left[\phi_\downarrow\right] \exp\left\{-S\left[\phi_\uparrow^*, \phi_\uparrow, \phi_\downarrow^*, \phi_\downarrow\right]/\hbar\right\} , \tag{1}$$

where the action is given by

$$\begin{aligned} S\left[\phi_\uparrow^*, \phi_\uparrow, \phi_\downarrow^*, \phi_\downarrow\right] &= \int_0^{\hbar\beta} d\tau \int d\mathbf{x}\phi_\uparrow^*(\mathbf{x},\tau)\left[\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2\nabla^2}{2m} - \mu\right]\phi_\uparrow(\mathbf{x},\tau) \\ &+ \int_0^{\hbar\beta} d\tau \int d\mathbf{x}\phi_\downarrow^*(\mathbf{x},\tau)\left[\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2\nabla^2}{2m} - \mu\right]\phi_\downarrow(\mathbf{x},\tau) \\ &+ V_0\int_0^{\hbar\beta} d\tau \int d\mathbf{x}\phi_\uparrow^*(\mathbf{x},\tau)\phi_\downarrow^*(\mathbf{x},\tau)\phi_\downarrow(\mathbf{x},\tau)\phi_\uparrow(\mathbf{x},\tau) . \end{aligned} \tag{2}$$

Upon lowering temperature, this system can undergo a phase transition corresponding to pseudo-spin ferromagnetism, in which, at zero temperature and for sufficiently strong interactions, all electrons have the pseudospin state $\left[|\uparrow\rangle + e^{i\theta}|\downarrow\rangle\right]/\sqrt{2}$, with $e^{i\theta}$ an arbitrary phase factor. Note that this corresponds to a pseudospin pointing in the $x-y$ plane (where $|\uparrow\rangle$ and $|\downarrow\rangle$ refer to the z-axis). Also note that this pseudospin state physically means that the electron is in a superposition of being in the top and being in the bottom layer.

a) (*5 points*) Argue, without calculations, why this system undergoes this phase transition. Why does the pseudospin ferromagnet never point in the z direction?

b) (*5 points*) Argue that a suitable order parameter for this phase transition is $\langle\Phi(\mathbf{x},\tau)\rangle = V_0\langle\phi_\uparrow^*(\mathbf{x},\tau)\phi_\downarrow(\mathbf{x},\tau)\rangle$. (The factor V_0 is convention.)

c) *(5 points)* Perform a Hubbard-Stratonovich transformation to the complex bosonic field $\Phi(\mathbf{x},\tau)$ that has the property that $\langle\Phi(\mathbf{x},\tau)\rangle = V_0\langle\phi_\uparrow^*(\mathbf{x},\tau)\phi_\downarrow(\mathbf{x},\tau)\rangle$, and show that the expression for Z after the Hubbard-Stratonovich transformation is written as

$$Z = \int d[\Phi^*]d[\Phi]d\left[\phi_\uparrow^*\right]d\left[\phi_\uparrow\right]d\left[\phi_\downarrow^*\right]d\left[\phi_\downarrow\right]\exp\left\{-S\left[\phi_\uparrow^*,\phi_\uparrow,\phi_\downarrow^*,\phi_\downarrow,\Phi^*,\Phi\right]/\hbar\right\},$$

with the action

$$\begin{aligned} S\left[\phi_\uparrow^*,\phi_\uparrow,\phi_\downarrow^*,\phi_\downarrow,\Phi^*,\Phi\right] &= \int_0^{\hbar\beta}d\tau\int d\mathbf{x}\phi_\uparrow^*(\mathbf{x},\tau)\left[\hbar\frac{\partial}{\partial\tau}-\frac{\hbar^2\nabla^2}{2m}-\mu\right]\phi_\uparrow(\mathbf{x},\tau) \\ &+ \int_0^{\hbar\beta}d\tau\int d\mathbf{x}\phi_\downarrow^*(\mathbf{x},\tau)\left[\hbar\frac{\partial}{\partial\tau}-\frac{\hbar^2\nabla^2}{2m}-\mu\right]\phi_\downarrow(\mathbf{x},\tau) \\ &+ \int_0^{\hbar\beta}d\tau\int d\mathbf{x}\left[\frac{|\Phi(\mathbf{x},\tau)|^2}{V_0}-\Phi^*(\mathbf{x},\tau)\phi_\uparrow^*(\mathbf{x},\tau)\phi_\downarrow(\mathbf{x},\tau)-\phi_\downarrow^*(\mathbf{x},\tau)\phi_\uparrow(\mathbf{x},\tau)\Phi(\mathbf{x},\tau)\right]. \end{aligned} \tag{3}$$

d) *(10 points)* Perform the functional integration over the electron fields $\phi_\uparrow^*,\phi_\uparrow,\phi_\downarrow^*,\phi_\downarrow$, so that the expression for Z is written as

$$Z = \int d[\Phi^*]d[\Phi]\exp\left\{-S_{\text{eff}}[\Phi^*,\Phi]/\hbar\right\},$$

and give the exact but formal expression for the effective action $S_{\text{eff}}[\Phi^*,\Phi]$.

e) *(10 points)* Calculate the effective action up to second order in the fields $\Phi^*(\mathbf{x},\tau)$ and $\Phi(\mathbf{x},\tau)$ and show that (up to irrelevant constants)

$$\begin{aligned} S_{\text{eff}}[\Phi^*,\Phi] &= \int_0^{\hbar\beta}d\tau\int d\mathbf{x}\int_0^{\hbar\beta}d\tau'\int d\mathbf{x}'\Phi^*(\mathbf{x},\tau)\left[\frac{1}{V_0}\delta(\mathbf{x}-\mathbf{x}')\delta(\tau-\tau')\right. \\ &\left.+\frac{1}{\hbar}G_0(\mathbf{x}-\mathbf{x}';\tau-\tau')G_0(\mathbf{x}'-\mathbf{x};\tau'-\tau)\right]\Phi(\mathbf{x}',\tau')+\mathcal{O}\left(|\Phi|^4\right). \end{aligned} \tag{4}$$

NB: You do not have to show that the terms beyond second order start at order $|\Phi|^4$.

f) *(10 points)* To describe the phase transition we consider field configurations Φ_0 independent of position and imaginary time $\Phi(\mathbf{x},\tau)=\Phi_0$. The effective action is then up to second order given by $S_{\text{eff}}[\Phi_0^*,\Phi_0]\equiv\hbar\beta A f_L(|\Phi_0|)$ with $f_L(|\Phi_0|)=\alpha(T)|\Phi_0|^2$, with $f_L(|\Phi_0|)$ the Landau free energy density that describes the phase transition. Show that $\alpha(T)$ is given by

$$\alpha(T) = \frac{1}{V_0}+\frac{1}{A}\sum_{\mathbf{k}}N_F'(\epsilon_{\mathbf{k}}-\mu),$$

where A is the area of the two-dimension electron gases, $\epsilon_{\mathbf{k}}=\hbar^2\mathbf{k}^2/(2m)$, and $N_F(x)=[e^{\beta x}+1]^{-1}$ is the Fermi-Dirac distribution function, with $N_F'(x)=dN_F(x)/dx$ its derivative.

g) *(5 points)* Assume that the phase transition is continuous. Explain the criterion for the critical temperature and give the equation that determines the critical temperature T_c, without solving it.

h) *(10 points)* Calculate the zero-temperature limit of $\alpha(T)$, and show from this that the interaction strength V_0 needs to be larger than a critical interaction strength $V_{0,c}$ in order for the system to become pseudospin polarized. Give this critical interaction strength.

It turns out that the above system can be mapped to a different system, where the pseudospin ferromagnetic phase transition becomes a BCS transition. In order to see this, we perform a so-called particle-hole transformation on the bottom layer, that is, we make a variable substitution in the path integral to new fields $\tilde{\phi}^*_\downarrow, \tilde{\phi}_\downarrow$. The substitution is $\tilde{\phi}^*_\downarrow(\mathbf{x},\tau) = \phi_\downarrow(\mathbf{x},\tau)$, and $\tilde{\phi}_\downarrow(\mathbf{x},\tau) = \phi^*_\downarrow(\mathbf{x},\tau)$. The new fields are also fermionic (Grassmann) fields and the measure of the path integral is invariant. Note that in the operator language the creation operator corresponding to the field $\tilde{\phi}^*(\mathbf{x},\tau)$ destroys an electron, and therefore creates, by definition, a hole.

i) *(5 points)* Show that the action after this transformation becomes

$$\begin{aligned}
S\left[\phi^*_\uparrow,\phi_\uparrow,\tilde{\phi}^*_\downarrow,\tilde{\phi}_\downarrow\right] &= \int_0^{\hbar\beta} d\tau \int d\mathbf{x}\phi^*_\uparrow(\mathbf{x},\tau)\left[\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2\nabla^2}{2m} - \mu\right]\phi_\uparrow(\mathbf{x},\tau) \\
&+ \int_0^{\hbar\beta} d\tau \int d\mathbf{x}\tilde{\phi}^*_\downarrow(\mathbf{x},\tau)\left[\hbar\frac{\partial}{\partial\tau} + \frac{\hbar^2\nabla^2}{2m} + \mu\right]\tilde{\phi}_\downarrow(\mathbf{x},\tau) \\
&- V_0\int_0^{\hbar\beta} d\tau \int d\mathbf{x}\phi^*_\uparrow(\mathbf{x},\tau)\tilde{\phi}^*_\downarrow(\mathbf{x},\tau)\tilde{\phi}_\downarrow(\mathbf{x},\tau)\phi_\uparrow(\mathbf{x},\tau)\ .
\end{aligned} \tag{5}$$

We see that this action is identical to the action for spin one-half fermions that attract each other and we therefore expect a BCS transition, i.e., a Bose-Einstein condensation of fermionic pairs. The only difference with the BCS transition discussed in the course is that the spin down fermion has a dispersion relation $\mu - \epsilon_{\mathbf{k}}$, whereas the spin up fermion has the dispersion relation $\epsilon_{\mathbf{k}} - \mu$.

j) *(5 points)* Given this observation, generalize Eq.(12.20) from the book to this case, and show that you obtain from this the same equation for T_c as in part h).

Linear-response theory and Bose-Einstein condensation *(30 points)*

Consider a Landau free energy density for a phase transition characterized by an order parameter that is a complex number, denoted by ϕ_0. Assume that there are sources j and j^*, for example external fields, that couple to the order parameter, so that the total Landau free energy becomes

$$f_L(|\phi_0|) = \alpha(T)|\phi_0|^2 + \frac{\beta_0}{2}|\phi_0|^4 - j^*\phi_0 - \phi_0^* j\ ,$$

where $\beta_0 > 0$ and $\alpha(T) = \alpha_0(T - T_c)$ changes sign at a critical temperature T_c (NB: $\alpha_0 > 0$). We define the response function Π according to

$$\langle\phi_0\rangle = \Pi j,$$

for $j \to 0$ and $T \geq T_c$.

a) *(5 points)* Show that $\Pi \propto 1/\alpha(T)$, so that the phase transition is signaled by a diverging response function Π.

For the remainder of this exercise we consider a homogeneous system of interacting spinless bosons. The action describing the system is $S = S_0 + S_{\text{int}}$, with

$$S_0[\phi^*,\phi] = \int_0^{\hbar\beta} d\tau \int d\mathbf{x}\phi^*(\mathbf{x},\tau)\left[\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2\boldsymbol{\nabla}^2}{2m} - \mu\right]\phi(\mathbf{x},\tau)\ ,$$

and

$$S_{\text{int}}[\phi^*,\phi] = \frac{1}{2}\int_0^{\hbar\beta} d\tau \int d\mathbf{x} \int d\mathbf{x}'\phi^*(\mathbf{x},\tau)\phi^*(\mathbf{x}',\tau)V(\mathbf{x}-\mathbf{x}')\phi(\mathbf{x}',\tau)\phi(\mathbf{x},\tau)\ .$$

Furthermore, we couple the system to complex-valued sources $I^*(\mathbf{x},\tau), I(\mathbf{x},\tau)$. This coupling is described by the action

$$S_I[\phi^*,\phi] = \int_0^{\hbar\beta} d\tau \int d\mathbf{x}\,[I^*(\mathbf{x},\tau)\phi(\mathbf{x},\tau) + \phi^*(\mathbf{x},\tau)I(\mathbf{x},\tau)] \ .$$

Sources of this kind can be implemented experimentally, for example in photo-emission spectroscopy.

Next, we define the following averages

$$\begin{aligned}
\langle\cdots\rangle_0 &\equiv \frac{1}{Z_0}\int d[\phi^*]d[\phi]\cdots e^{-S_0[\phi^*,\phi]/\hbar} \ , \\
\langle\cdots\rangle &\equiv \frac{1}{Z}\int d[\phi^*]d[\phi]\cdots e^{-\{S_0[\phi^*,\phi]+S_{\rm int}[\phi^*,\phi]\}/\hbar} \ , \\
\langle\cdots\rangle_I &\equiv \frac{1}{Z_I}\int d[\phi^*]d[\phi]\cdots e^{-\{S_0[\phi^*,\phi]+S_{\rm int}[\phi^*,\phi]+S_I[\phi^*,\phi]\}/\hbar} \ ,
\end{aligned}$$

where Z_0, Z, Z_I are the partition function of the non-interacting and interacting systems, and the system coupled to sources, respectively.

b) (*10 points*) Show that, to first order in $I^*(\mathbf{x},\tau)$ and $I(\mathbf{x},\tau)$,

$$\langle\phi(\mathbf{x},\tau)\rangle_I = -\frac{1}{\hbar}\int_0^{\hbar\beta} d\tau' \int d\mathbf{x}' \langle\phi(\mathbf{x},\tau)\phi^*(\mathbf{x}',\tau')\rangle I(\mathbf{x}',\tau') \equiv \frac{1}{\hbar}\int_0^{\hbar\beta} d\tau' \int d\mathbf{x}' G(\mathbf{x},\tau;\mathbf{x}',\tau') I(\mathbf{x}',\tau') \ ,$$

with $G(\mathbf{x},\tau;\mathbf{x}',\tau')$ the Green's function of the interacting system.

c) (*5 points*) Show that part a) and b) imply that a criterion for Bose-Einstein condensation is $1/G(\mathbf{0},0) = 0$, where $G(\mathbf{k}, i\omega_n)$ is the Fourier transform of the Green's function in the normal state.

d) (*5 points*) Show that for the non-interacting case this reduces to the well-known criterion for Bose-Einstein condensation.

e) (*5 points*) Show that for an interacting system the criterion for Bose-Einstein becomes $\mu = \hbar\Sigma(\mathbf{0},0)$, where $\hbar\Sigma(\mathbf{k},i\omega_n)$ is the Fourier transform of the self-energy. Consider contact interactions $V(\mathbf{x}-\mathbf{x}') = V_0\delta(\mathbf{x}-\mathbf{x}')$ and give the condition for Bose-Einstein condensation within the Hartree-Fock approximation.

The equations below hold for an interaction potential $V(\vec{x}-\vec{x}') = V_0\, \delta(\vec{x}-\vec{x}')$

given by

$$
\begin{aligned}
\alpha(T) &= -\frac{1}{V_0} - \frac{1}{\hbar^2 \beta V} \int_0^{\hbar\beta} \mathrm{d}\tau\, \mathrm{d}\tau' \int \mathrm{d}\mathbf{x}\, \mathrm{d}\mathbf{x}'\, G_{0;\uparrow}(\mathbf{x},\tau;\mathbf{x}',\tau') G_{0;\downarrow}(\mathbf{x},\tau;\mathbf{x}',\tau') \\
&= -\frac{1}{V_0} - \frac{1}{\hbar^2 \beta V} \sum_{n,n'} \sum_{\mathbf{k},\mathbf{k}'} G_{0;\uparrow}(\mathbf{k},\mathrm{i}\omega_n) G_{0;\downarrow}(\mathbf{k}',\mathrm{i}\omega_{n'}) \delta_{\mathbf{k},-\mathbf{k}'} \delta_{n,-n'} \\
&= -\frac{1}{V_0} - \frac{1}{\hbar^2 \beta V} \sum_n \sum_{\mathbf{k}} \frac{-\hbar}{-\mathrm{i}\hbar\omega_n + \varepsilon_{\mathbf{k}} - \mu} \frac{-\hbar}{\mathrm{i}\hbar\omega_n + \varepsilon_{\mathbf{k}} - \mu},
\end{aligned} \tag{12.20}
$$

where we substituted the Fourier expansion for the homogeneous noninteracting Green's functions, as discussed in Example 7.3, after which the integrals over position and imaginary time give rise to the Kronecker deltas $\delta_{\mathbf{k},-\mathbf{k}'}$ and $\delta_{n,-n'}$. Next, we split the fraction and perform the sum over Matsubara frequencies, giving

$$
\begin{aligned}
\alpha(T) &= -\frac{1}{V_0} - \frac{1}{\hbar^2 \beta V} \sum_{\mathbf{k}} \sum_n \frac{-\hbar}{2(\varepsilon_{\mathbf{k}} - \mu)} \left\{ \frac{-\hbar e^{\mathrm{i}\omega_n \eta}}{-\mathrm{i}\hbar\omega_n + \varepsilon_{\mathbf{k}} - \mu} + \frac{-\hbar e^{\mathrm{i}\omega_n \eta}}{\mathrm{i}\hbar\omega_n + \varepsilon_{\mathbf{k}} - \mu} \right\} \\
&= -\frac{1}{V_0} - \frac{1}{V} \sum_{\mathbf{k}} \frac{1}{2(\varepsilon_{\mathbf{k}} - \mu)} \left\{ 1 - \frac{2}{e^{\beta(\varepsilon_{\mathbf{k}} - \mu)} + 1} \right\},
\end{aligned} \tag{12.21}
$$

where we used (7.31) and the result from Exercise 7.2. As explained in Sect. 10.4, the interaction parameter V_0 is related to the experimentally known s-wave scattering length a. Using (10.54), we finally arrive at [81]

$$
\begin{aligned}
\alpha(T) &= -\frac{m}{4\pi a \hbar^2} - \frac{1}{V} \sum_{\mathbf{k}} \left\{ \frac{1}{2(\varepsilon_{\mathbf{k}} - \mu)} \left(1 - \frac{2}{e^{\beta(\varepsilon_{\mathbf{k}} - \mu)} + 1} \right) - \frac{1}{2\varepsilon_{\mathbf{k}}} \right\} \\
&= -\frac{m}{4\pi a \hbar^2} - \frac{1}{V} \sum_{\mathbf{k}} \left[\frac{\tanh\left(\beta(\varepsilon_{\mathbf{k}} - \mu)/2\right)}{2(\varepsilon_{\mathbf{k}} - \mu)} - \frac{1}{2\varepsilon_{\mathbf{k}}} \right].
\end{aligned} \tag{12.22}
$$

12.5 Critical Temperature

As explained in Chap. 9, the second-order coefficient $\alpha(T)$ of the Landau free energy $f_{\mathrm{L}}(|\Delta|)$ determines the critical temperature for a second-order phase transition. It is this term that changes sign at the critical temperature, such that the minimum of the Landau free energy shifts away from zero, yielding a nonzero order parameter $\langle\Delta\rangle$. As a result, the critical temperature $k_{\mathrm{B}}T_{\mathrm{c}} \equiv 1/\beta_{\mathrm{c}}$ is determined by the condition $\alpha(k_{\mathrm{B}}T_{\mathrm{c}}) = 0$. As we now show, in the weakly-interacting limit when the critical temperature is low, we can obtain an analytic expression for the critical temperature from BCS theory. We start with converting the sum on the right-hand side of (12.22) into an integral such that it becomes