INSTITUTE FOR THEORETICAL PHYSICS
UTRECHT UNIVERSITY

NS-TP402M

Midterm Exam Statistical Field Theory

Tuesday, 3 November, 2009, 14:00-16:00

1. *Use a seperate sheet for every excercise.*
2. *Write your name and initials on all sheets, on the first sheet also your address and your student ID number.*
3. *Write clearly, unreadable work cannot be corrected.*
4. *You may use your lecture notes and the book by Stoof et al.*
5. *Distribute your time evenly between the two exercises.*

Is a pair of fermions a boson?

We consider an ideal gas of spin one-half fermions without external potential in a box of volume V. The second-quantized hamiltonian is given by

$$\hat{H} = \int d\mathbf{x}\hat{\psi}^{\dagger}_{\uparrow}(\mathbf{x}) \left[-\frac{\hbar^2\nabla^2}{2m}\right] \hat{\psi}_{\uparrow}(\mathbf{x}) + \int d\mathbf{x}\hat{\psi}^{\dagger}_{\downarrow}(\mathbf{x}) \left[-\frac{\hbar^2\nabla^2}{2m}\right] \hat{\psi}_{\downarrow}(\mathbf{x}) ,$$

where $\hat{\psi}_\sigma(\mathbf{x})$ annihilates a particle with spin state $|\sigma\rangle$, with $\sigma \in \{\uparrow, \downarrow\}$.

a) Give the anticommutation relations for the operators $\hat{\psi}_\sigma(\mathbf{x})$ and their hermitian conjugate $\hat{\psi}^{\dagger}_\sigma(\mathbf{x})$.

Because of the absence of an external potential it is convenient to perform a Fourier transform and write

$$\hat{\psi}_\sigma(\mathbf{x}) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} \hat{c}_{\mathbf{k},\sigma} e^{i\mathbf{k}\cdot\mathbf{x}} .$$

b) Show that the anticommutator $[c_{\mathbf{k},\sigma}, c^{\dagger}_{\mathbf{k}',\sigma'}]_+ = \delta_{\mathbf{k}\mathbf{k}'}\delta_{\sigma\sigma'}$

c) Rewrite the hamiltonian in terms of $\hat{c}_{\mathbf{k},\sigma}$ and $\hat{c}^{\dagger}_{\mathbf{k},\sigma}$.

We consider the system in the grand-canonical ensemble and use that the partition function is written as a path integral over all anti-periodic Grassmann field evolutions as

$$Z = \int d[\phi^*]d[\phi]e^{-S[\phi^*,\phi]/\hbar} ,$$

where

$$S[\phi^*, \phi] = \int_0^{\hbar\beta} d\tau \int d\mathbf{x} \sum_{\sigma\in\{\uparrow,\downarrow\}} \phi^*_\sigma(\mathbf{x},\tau) \left[\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2\nabla^2}{2m} - \mu\right] \phi_\sigma(\mathbf{x},\tau) ,$$

and where the field $\phi_\sigma(\mathbf{x},\tau)$ corresponds to the operator $\hat{\psi}_\sigma(\mathbf{x})$. The Green's function $G_{\sigma\sigma'}(\mathbf{x},\tau;\mathbf{x}',\tau')$ is in this case defined by

$$S[\phi^*, \phi] \equiv \int_0^{\hbar\beta} d\tau \int d\mathbf{x} \int_0^{\hbar\beta} d\tau' \int d\mathbf{x}' \sum_{\sigma,\sigma'\in\{\uparrow,\downarrow\}} \phi^*_\sigma(\mathbf{x},\tau)[-\hbar G^{-1}_{\sigma\sigma'}(\mathbf{x},\tau;\mathbf{x}',\tau')]\phi_{\sigma'}(\mathbf{x}',\tau') .$$

d) Give the equation for $G^{-1}_{\sigma\sigma'}(\mathbf{x},\tau;\mathbf{x}',\tau')$ that follows from the above, and derive from it the equation for $G_{\sigma\sigma'}(\mathbf{x},\tau;\mathbf{x}',\tau')$. Show that the latter is solved by

$$G_{\sigma\sigma'}(\mathbf{x},\tau;\mathbf{x}',\tau') = \delta_{\sigma\sigma'}\frac{1}{\hbar\beta V}\sum_{\mathbf{k},n}\frac{-\hbar}{-i\hbar\omega_n+\epsilon_{\mathbf{k}}-\mu}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')-i\omega_n(\tau-\tau')} ,$$

with ω_n the fermionic Matsubara frequencies.

We consider now the operator $\hat{\Delta}(\mathbf{x}) = \hat{\psi}_\uparrow(\mathbf{x})\hat{\psi}_\downarrow(\mathbf{x})$, and its Fourier transform $\hat{b}_{\mathbf{k}}$ determined from

$$\hat{\Delta}(\mathbf{x}) = \frac{1}{V}\sum_{\mathbf{k}}\hat{b}_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}} .$$

e) Show that $\hat{b}_{\mathbf{k}} = \sum_{\mathbf{k}'}\hat{c}_{\mathbf{k}',\uparrow}\hat{c}_{\mathbf{k}-\mathbf{k}',\downarrow}$.

f) Show that the commutator

$$[\hat{b}_{\mathbf{k}},\hat{b}^\dagger_{\mathbf{k}'}]_- = \delta_{\mathbf{k}\mathbf{k}'} - \sum_{\mathbf{q},\sigma\in\{\uparrow,\downarrow\}}\hat{c}^\dagger_{\mathbf{k}'-\mathbf{q},\sigma}\hat{c}_{\mathbf{k}-\mathbf{q},\sigma} .$$

g) Using this result, answer the question in the title of this exercise.

We now introduce the field corresponding to the operator $\hat{\Delta}(\mathbf{x})$ as $\Delta(\mathbf{x},\tau) = \phi_\uparrow(\mathbf{x},\tau)\phi_\downarrow(\mathbf{x},\tau)$. The Green's function for this operator is defined as $G_\Delta(\mathbf{x},\tau;\mathbf{x}',\tau') = -\langle\Delta(\mathbf{x},\tau)\Delta^*(\mathbf{x}',\tau')\rangle$.

h) Show that $G_\Delta(\mathbf{x},\tau;\mathbf{x}',\tau') = -G_{\uparrow\uparrow}(\mathbf{x},\tau;\mathbf{x}',\tau')G_{\downarrow\downarrow}(\mathbf{x},\tau;\mathbf{x}',\tau')$.

i) Show that its fourier transform $G_\Delta(\mathbf{k},i\Omega_n)$ is given by

$$G_\Delta(\mathbf{k},i\Omega_n) = \frac{1}{V}\sum_{\mathbf{k}'}\frac{1-N_{\rm FD}(\epsilon_{\mathbf{k}+\mathbf{k}'})-N_{\rm FD}(\epsilon_{\mathbf{k}'})}{i\hbar\Omega_n-\epsilon_{\mathbf{k}+\mathbf{k}'}-\epsilon_{\mathbf{k}'}+2\mu} ,$$

with $N_{\rm FD}(\epsilon) = [\exp\{\beta(\epsilon-\mu)\}+1]^{-1}$ the Fermi-Dirac distribution function and $\epsilon_{\mathbf{k}} = \hbar^2\mathbf{k}^2/(2m)$.

j) Is Ω_n a bosonic or fermionic Matsubara frequency?