Institute for Theoretical Physics
Utrecht University

Final Exam Statistical Field Theory (NS-TP402M)

Tuesday, 31 January, 2012, 15:00-18:00

1. *Write your name and initials on all sheets, on the first sheet also your student ID number.*
2. *Write clearly, unreadable work cannot be corrected.*
3. *You may use your notes and the book by Stoof et al.*
4. *Distribute your time evenly over all exercises, don't spend an enormous amount of time on correcting minus signs, factors of two and/or π, etc. In case you suspect that you have made a calculational error but don't have enough time to correct it, point it out in words.*
5. *Give clear motivation, argumentation, and explanation with each answer.*

Exercise 1: BCS-like transition in a Bose gas (*70 points*)

In this exercise we consider interacting spinless bosons. Upon lowering the temperature to a critical temperature T_c, such a system may initially undergo a phase transition to a pair-condensed BCS-like state where the bosonic atoms form pairs that Bose-Einstein condense, before (upon further lowering the temperature to a critical value T_{BEC}), undergoing conventional Bose-Einstein condensation. That is, for temperatures $T_c > T > T_{BEC}$ we have that $\langle\Delta(\mathbf{x},\tau)\rangle \neq 0$, whereas $\langle\phi(\mathbf{x},\tau)\rangle = 0$. For temperatures $T < T_{BEC}$ we have that the order parameter for Bose-Einstein condensation becomes nonzero, i.e., $\langle\phi(\mathbf{x},\tau)\rangle \neq 0$. Note that this necessarily implies that the expectation value of the order parameter for the BCS-like transition,

$$\Delta(\mathbf{x},\tau) = V_0\phi(\mathbf{x},\tau)\phi(\mathbf{x},\tau) \ , \tag{1}$$

remains nonzero for $T < T_{BEC} < T_c$.

We neglect any external potential and consider a three-dimensional homogeneous system of volume V. The starting point is the grand-canonical partition function given by

$$Z = \int d[\phi^*]\, d[\phi]\, d \exp\{-S[\phi^*,\phi]/\hbar\} \ , \tag{2}$$

where the action is given by

$$S[\phi^*,\phi] \ = \ \int_0^{\hbar\beta} d\tau \int d\mathbf{x}\phi^*(\mathbf{x},\tau)\left[\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2\nabla^2}{2m} - \mu + \frac{V_0}{2}|\phi(\mathbf{x},\tau)|^2\right]\phi(\mathbf{x},\tau) \ . \tag{3}$$

Note that, in the above, we have approximated the interatomic interaction with $V(\mathbf{x}-\mathbf{x}') \simeq V_0\delta(\mathbf{x}-\mathbf{x}')$. Furthermore, the mass of the atoms is m, and their chemical potential μ.

a) (*5 points*) For which sign of V_0 do you expect a transition to a pair-condensed state?

b) (*10 points*) Decouple the interaction with a Hubbard-Stratonovich transformation to the pair field $\Delta(\mathbf{x},\tau)$ and show that after the transformation we have that

$$Z = \int d[\Delta^*]d[\Delta]d[\phi^*]\, d[\phi] \exp\{-S[\phi^*,\phi,\Delta^*,\Delta]/\hbar\} \ .$$

with the action

$$S[\phi^*,\phi,\Delta^*,\Delta] = \int_0^{\hbar\beta} d\tau \int d\mathbf{x}\phi^*(\mathbf{x},\tau)\left[\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2\nabla^2}{2m} - \mu\right]\phi(\mathbf{x},\tau)$$
$$+ \frac{1}{2}\int_0^{\hbar\beta} d\tau \int d\mathbf{x}\left[\Delta^*(\mathbf{x},\tau)\phi(\mathbf{x},\tau)\phi(\mathbf{x},\tau) + \phi^*(\mathbf{x},\tau)\phi^*(\mathbf{x},\tau)\Delta(\mathbf{x},\tau) - \frac{|\Delta(\mathbf{x},\tau)|^2}{V_0}\right] . \quad (4)$$

c) *(10 points)* Perform the functional integration over the Bose fields ϕ^*, ϕ, so that the expression for Z is written as

$$Z = \int d[\Delta^*]d[\Delta]\exp\{-S_{\text{eff}}[\Delta^*,\Delta]/\hbar\} ,$$

and give the exact but formal expression for the effective action $S_{\text{eff}}[\Delta^*,\Delta]$.

d) *(10 points)* Calculate the effective action up to second order in the field $\Delta(x,\tau)$ and show that (up to irrelevant constants)

$$S_{\text{eff}}[\Delta^*,\Delta] = -\int_0^{\hbar\beta} d\tau \int d\mathbf{x} \int_0^{\hbar\beta} d\tau' \int d\mathbf{x}'\Delta^*(\mathbf{x},\tau)\left[\frac{1}{2V_0}\delta(\mathbf{x}-\mathbf{x}')\delta(\tau-\tau')\right.$$
$$\left.+\frac{1}{2\hbar}G_0(\mathbf{x}-\mathbf{x}';\tau-\tau')G_0(\mathbf{x}-\mathbf{x}';\tau-\tau')\right]\Delta(\mathbf{x}',\tau') + \mathcal{O}\left(|\Delta|^4\right) . \quad (5)$$

NB: You do not have to show that the terms beyond second order start at order $|\Delta|^4$.

e) *(10 points)* To describe the ferromagnetic phase transition we consider field configurations Δ_0 independent of position and imaginary time $\Delta(\mathbf{x},\tau) = \Delta_0$. The effective action is then up to second order given by $S_{\text{eff}}[\Delta_0^*,\Delta_0] \equiv \hbar\beta V f_L(|\Delta_0|)$ with $f_L(|\Delta_0|) = \alpha(T)|\Delta_0|^2$, with $f_L(|\Delta_0|)$ the Landau free energy density that describes the phase transition. Show that $\alpha(T)$ is given by

$$\alpha(T) = -\frac{1}{2V_0} - \frac{1}{2V}\sum_{\mathbf{k}}\frac{1+2N_B(\epsilon_{\mathbf{k}}-\mu)}{2(\epsilon_{\mathbf{k}}-\mu)} ,$$

where $\epsilon_{\mathbf{k}} = \hbar^2\mathbf{k}^2/(2m)$, and $N_B(x) = [e^{\beta x}-1]^{-1}$ (with $\beta = 1/k_BT$), is the Bose-Einstein distribution function.

f) *(5 points)* Assume that the phase transition is continuous. Explain the criterion for the critical temperature and give the equation that determines the critical temperature T_c, without solving it.

g) *(5 points)* Ignore the ultraviolet divergence in the above equation for $\alpha(T)$ which can be absorbed into a redefinition of V_0. That is, from now on take

$$\alpha(T) = -\frac{1}{2V_0} - \frac{1}{2V}\sum_{\mathbf{k}}\frac{N_B(\epsilon_{\mathbf{k}}-\mu)}{(\epsilon_{\mathbf{k}}-\mu)} .$$

Using this equation, discuss under which condition for V_0 a transition to a pair-condensed state occurs.

h) *(15 points)* Consider now the system at temperatures $T_{BEC} \leq T < T_c$, and assume that $\Delta(\mathbf{x},\tau)$ can be treated in mean-field approximation, so that we have $\Delta(\mathbf{x},\tau) = \Delta_0$, and that $\Delta_0 = \Delta_0(T)$ is known as a function of temperature. Derive the dispersion relation for the atoms in this temperature range, and argue, based on this dispersion relation, that the critical temperature for Bose-Einstein condensation is determined by $\mu(T_{BEC}) = |\Delta_0(T_{BEC})|$.

Exercise 2: Electron-magnon coupling (*30 points*)

An electron in a conducting ferromagnet interacts with fluctuations in the magnetization. In particular, an electron can excite wave-like vibrations of the magnetization, i.e., spin waves that, once quantized, are also referred to as magnons. An effective action that describes the electron-magnon system is given by

$$\begin{aligned} S\left[c^*_\uparrow, c_\uparrow, c^*_\downarrow, c_\downarrow, b^*, b\right] &= \int_0^{\hbar\beta} d\tau \sum_{\mathbf{k},\sigma\in\{\uparrow,\downarrow\}} c^*_{\mathbf{k},\sigma}(\tau)\left[\hbar\frac{\partial}{\partial\tau} + \epsilon_{\mathbf{k}} - \frac{\sigma\Delta}{2} - \mu\right] c_{\mathbf{k},\sigma}(\tau) + \int_0^{\hbar\beta} d\tau \sum_{\mathbf{k}} b^*_{\mathbf{k}}(\tau)\left[\hbar\frac{\partial}{\partial\tau} + \hbar\omega_{\mathbf{k}}\right] b_{\mathbf{k}}(\tau) \\ &+ \int_0^{\hbar\beta} d\tau \left[\sum_{\mathbf{k}}\sum_{\mathbf{q}} t(\mathbf{q}) b^*_{\mathbf{q}}(\tau) c^*_{\mathbf{k}-\mathbf{q},\downarrow}(\tau) c_{\mathbf{k},\uparrow}(\tau) + \sum_{\mathbf{k}}\sum_{\mathbf{q}} t(\mathbf{q}) b_{\mathbf{q}}(\tau) c^*_{\mathbf{k},\uparrow}(\tau) c_{\mathbf{k}-\mathbf{q},\downarrow}(\tau)\right] , \end{aligned} \tag{6}$$

where μ is the chemical potential for the electrons and Δ the so-called exchange splitting in the ferromagnet. The field $c_{\mathbf{k},\sigma}(\tau)$ describes the electrons with spin projection $\sigma \in \{\uparrow,\downarrow\}$, and the field $b_{\mathbf{k}}(\tau)$ describes the magnons. The magnon-magnon interaction and electron-electron interaction have been neglected. The matrix elements that determine the coupling between electrons and magnons are denoted by $t(\mathbf{q})$ and assumed to be real. Furthermore, $\epsilon_{\mathbf{k}}$ and $\hbar\omega_{\mathbf{k}}$ are the dispersion relations for electrons and magnons, respectively.

a) (*5 points*) Give the physical meaning of the last two terms in the action, paying also attention to the electron spin in the electron-magnon coupling.

b) (*15 points*) Derive an effective action for the electrons by performing the Gaussian integral over the magnon field, and show that this leads to an interaction term for the electrons that is of the form

$$\int_0^{\hbar\beta} d\tau \int d\mathbf{x} \int_0^{\hbar\beta} d\tau' \int d\mathbf{x}'\ \phi^*_\uparrow(\mathbf{x}',\tau')\phi^*_\downarrow(\mathbf{x},\tau) V(\mathbf{x}-\mathbf{x}';\tau-\tau')\phi_\downarrow(\mathbf{x}',\tau')\phi_\uparrow(\mathbf{x},\tau) ,$$

where the electron field $\phi_\sigma(\mathbf{x},\tau)$ is the Fourier transform of $c_{\mathbf{k},\sigma}(\tau)$.

c) (*5 points*) Give the expression for $V(\mathbf{x}-\mathbf{x}';\tau-\tau')$ as a Fourier expansion, i.e., without working out the sums over momenta and Matsubara frequencies.

d) (*5 points*) Draw the Feynman diagram corresponding to this interaction, indicating also the electron spin.