INSTITUTE FOR THEORETICAL PHYSICS
UTRECHT UNIVERSITY

# Midterm Exam Statistical Field Theory

Tuesday, 9 November, 2010, 14:00-16:00

1. *Use a seperate sheet for every excercise.*
2. *Write your name and initials on all sheets, on the first sheet also your address and your student ID number.*
3. *Write clearly, unreadable work cannot be corrected.*
4. *You may use your lecture notes and the book by Stoof* et al.
5. *Distribute your time evenly between the two exercises.*

## Density fluctuations in a Bose gas

We consider an ideal gas of $N$ spinless bosons in an external potential $V^{\text{ex}}(\mathbf{x})$. The second-quantized hamiltonian is given by

$$\hat{H} = \int d\mathbf{x} \hat{\psi}^\dagger(\mathbf{x}) \left[ -\frac{\hbar^2 \nabla^2}{2m} + V^{\text{ex}}(\mathbf{x}) \right] \hat{\psi}(\mathbf{x}) ,$$

where $\hat{\psi}(\mathbf{x})$ annihilates a particle at position $\mathbf{x}$.

a) Give the commutation relations for the operator $\hat{\psi}(\mathbf{x})$ and its hermitian conjugate $\hat{\psi}^\dagger(\mathbf{x})$.

Because of the external potential it is convenient to expand the field operators in terms of (properly normalized) single-particle eigenstates $\chi_{\mathbf{n}}(\mathbf{x})$ of the single-particle hamiltonian according to

$$\hat{\psi}(\mathbf{x}) = \sum_{\mathbf{n}} \hat{c}_{\mathbf{n}} \chi_{\mathbf{n}}(\mathbf{x}) .$$

b) Show that the commutator $[c_{\mathbf{n}}, c^\dagger_{\mathbf{n}'}] = \delta_{\mathbf{n}\mathbf{n}'}$

c) Rewrite the hamiltonian in terms of $\hat{c}_{\mathbf{n}}$ and $\hat{c}^\dagger_{\mathbf{n}}$.

We consider the system in the grand-canonical ensemble and use that the partition function can be written as a path integral over all periodic complex field evolutions as

$$Z = \int d[\phi^*] d[\phi] e^{-S[\phi^*,\phi]/\hbar} ,$$

where

$$S[\phi^*,\phi] = \int_0^{\hbar\beta} d\tau \int d\mathbf{x} \phi^*(\mathbf{x},\tau) \left[ \hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2\nabla^2}{2m} + V^{\text{ex}}(\mathbf{x}) - \mu \right] \phi(\mathbf{x},\tau) ,$$

and where the field $\phi(\mathbf{x},\tau)$ corresponds to the operator $\hat{\psi}(\mathbf{x})$. The Green's function $G(\mathbf{x},\tau;\mathbf{x}',\tau')$ is in this case defined by

$$S[\phi^*,\phi] \equiv \int_0^{\hbar\beta} d\tau \int d\mathbf{x} \int_0^{\hbar\beta} d\tau' \int d\mathbf{x}' \phi^*(\mathbf{x},\tau)[-\hbar G^{-1}(\mathbf{x},\tau;\mathbf{x}',\tau')]\phi(\mathbf{x}',\tau') .$$

d) Give the equation for $G^{-1}(\mathbf{x},\tau;\mathbf{x}',\tau')$ that follows from the above, and derive from it the equation for $G(\mathbf{x},\tau;\mathbf{x}',\tau')$. Show that the latter is solved by

$$G(\mathbf{x},\tau;\mathbf{x}',\tau') = \frac{1}{\hbar\beta}\sum_{\mathbf{n},n} \frac{-\hbar}{-i\hbar\omega_n + \epsilon_{\mathbf{n}} - \mu} \chi_{\mathbf{n}}(\mathbf{x})\chi^*_{\mathbf{n}}(\mathbf{x}') e^{-i\omega_n(\tau-\tau')} ,$$

with $\omega_n$ the bosonic Matsubara frequencies, and $\epsilon_{\mathbf{n}}$ the energies of the single-particle eigenstates $\chi_{\mathbf{n}}(\mathbf{x})$.

e) Carry out the summation over Matsubara frequencies in the above expression for the Green's function and show that

$$G(\mathbf{x},\tau;\mathbf{x}',\tau') = -\sum_{\mathbf{n}} \left[1 + n_B(\epsilon_{\mathbf{n}} - \mu)\right] \chi_{\mathbf{n}}(\mathbf{x})\chi^*_{\mathbf{n}}(\mathbf{x}') e^{-(\epsilon_{\mathbf{n}}-\mu)(\tau-\tau')/\hbar} ,$$

if $\tau > \tau'$, and

$$G(\mathbf{x},\tau;\mathbf{x}',\tau') = -\sum_{\mathbf{n}} n_B(\epsilon_{\mathbf{n}} - \mu) \chi_{\mathbf{n}}(\mathbf{x})\chi^*_{\mathbf{n}}(\mathbf{x}') e^{-(\epsilon_{\mathbf{n}}-\mu)(\tau-\tau')/\hbar} ,$$

when $\tau' > \tau$. Here, $n_B(x) = [e^{\beta x} - 1]^{-1}$ is the Bose distribution function, with $\beta$ the inverse thermal energy.

f) Rederive this result with operator methods by starting from

$$G(\mathbf{x},\tau;\mathbf{x}',\tau') = -\left\langle T\left[\hat{\psi}(\mathbf{x},\tau)\hat{\psi}^\dagger(\mathbf{x}',\tau')\right]\right\rangle ,$$

where $T[\cdots]$ is the time-ordering operator on the imaginary-time axis, and $\hat{\psi}(\mathbf{x},\tau)$ and $\hat{\psi}^\dagger(\mathbf{x},\tau)$ are the grand-canonical imaginary-time Heisenberg operators corresponding to the operators $\hat{\psi}(\mathbf{x})$ and $\hat{\psi}^\dagger(\mathbf{x})$. The expectation value $\langle\cdots\rangle$ is taken in the grand-canonical ensemble, as usual. Evaluate this expectation value by first solving the Heisenberg equation of motion for the field operators by expanding the field operators in eigenstates $\chi_{\mathbf{n}}(\mathbf{x})$.

g) Consider the density operator $\hat{n}(\mathbf{x}) = \hat{\psi}^\dagger(\mathbf{x})\hat{\psi}(\mathbf{x})$ and show that its commutator is given by

$$[\hat{n}(\mathbf{x}),\hat{n}(\mathbf{x}')] = \delta(\mathbf{x}-\mathbf{x}')\left[\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}(\mathbf{x}') - \hat{\psi}^\dagger(\mathbf{x}')\hat{\psi}(\mathbf{x})\right] .$$

h) For the remainder of this exercise, we assume that the system is in its ground state. Argue that this state is given by

$$|\Psi\rangle = \frac{\left(\hat{c}^\dagger_{\mathbf{0}}\right)^N}{\sqrt{N!}}|0\rangle ,$$

where $\chi_{\mathbf{0}}(\mathbf{x})$ is the ground state of the single-particle potential.

i) Evaluate the expectation value of the operator $\hat{n}(\mathbf{x})\hat{n}(\mathbf{x}')$ for the state $|\Psi\rangle$. Is Wick's theorem obeyed for this expectation value?

j) Give the chemical potential in the limit that $N \to \infty$.