Institute for Theoretical Physics
Utrecht University

Midterm Exam Statistical Field Theory (NS-TP402M)

Tuesday, 8 November, 2011, 15:00-18:00

1. *Use a separate sheet for every exercise.*
2. *Write your name and initials on all sheets, on the first sheet also your student ID number.*
3. *Write clearly, unreadable work cannot be corrected.*
4. *You may use your notes, solutions to exercises, and the book by Stoof et al.*
5. *Distribute your time evenly over the exam, don't spend an enormous amount of time on correcting minus signs, factors of two and/or π, etc. In case you suspect that you have made a calculational error but don't have enough time to correct it, point it out in words.*
6. *Give the motivation, explanation, and calculations leading up to each answer and/or solution.*

Non-interacting spin-one-half Fermi gas in a magnetic field

We consider an ideal (i.e., non-interacting) homogeneous Fermi gas in three dimensions (volume V) in an external homogeneous magnetic field $\mathbf{B}$. The Euclidean action describing this system is given by

$$S[\phi^*,\phi] = \int_0^{\hbar\beta} d\tau \int d\mathbf{x} \sum_{\sigma,\sigma'\in\{\uparrow,\downarrow\}} \phi^*_\sigma(\mathbf{x},\tau) \left[\left(\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2\boldsymbol{\nabla}^2}{2m} - \mu \right) \delta_{\sigma,\sigma'} - \sum_{\alpha\in\{x,y,z\}} B_\alpha \tau^\alpha_{\sigma,\sigma'} \right] \phi_{\sigma'}(\mathbf{x},\tau) \ ,$$

where $\phi^*_\sigma(\mathbf{x},\tau)$ and $\phi_\sigma(\mathbf{x},\tau)$ are the usual Grassman-valued fields that are anti-periodic on the imaginary-time axis ranging from zero to $\hbar\beta = \hbar/k_BT$. In the above we have absorbed all pre-factors in the definition of the magnetic field. Furthermore, the τ^α are the Pauli matrices given by

$$\tau^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \tau^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \tau^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ . \tag{1}$$

We first take the magnetic field in the z-direction, i.e., $B_z = B$ and $B_x = B_y = 0$.

a) (*5 points*) Give the differential equation that determines the Green's function $G_{\sigma,\sigma'}(\mathbf{x},\tau;\mathbf{x}',\tau')$ defined by

$$S[\phi^*,\phi] = \int_0^{\hbar\beta} d\tau \int d\mathbf{x} \int_0^{\hbar\beta} d\tau' \int d\mathbf{x}' \phi^*_\sigma(\mathbf{x},\tau) \left[-\hbar G^{-1}_{\sigma,\sigma'}(\mathbf{x},\tau;\mathbf{x}',\tau') \right] \phi_{\sigma'}(\mathbf{x}',\tau') \ . \tag{2}$$

b) (*10 points*) Show that this equation is solved by

$$G_{\sigma,\sigma'}(\mathbf{x},\tau;\mathbf{x}',\tau') = \frac{1}{\hbar\beta V} \sum_{\mathbf{k},n} \frac{-\hbar}{-i\hbar\omega_n + \epsilon_{\mathbf{k}} - \sigma B - \mu} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}') - i\omega_n(\tau-\tau')} \delta_{\sigma,\sigma'} \ , \tag{3}$$

where the number $\sigma = +1$ if the index $\sigma =\uparrow$, and $\sigma = -1$ if the index $\sigma =\downarrow$, and $\epsilon_{\mathbf{k}} = \hbar^2\mathbf{k}^2/2m$

c) (*5 points*) The spin density $s_\alpha(\mathbf{x})$ (with $\alpha \in \{x,y,z\}$) is defined by

$$s_\alpha(\mathbf{x}) = \frac{\hbar}{2} \sum_{\sigma,\sigma'\in\{\uparrow,\downarrow\}} \langle \hat{\psi}^\dagger_\sigma(\mathbf{x}) \tau^\alpha_{\sigma,\sigma'} \hat{\psi}_{\sigma'}(\mathbf{x}) \rangle \ , \tag{4}$$

in terms of fermionic creation and annihilation operators $\hat{\psi}^\dagger_\sigma(\mathbf{x})$ and $\hat{\psi}_\sigma(\mathbf{x})$. Argue that in terms of the Green's function

$$s_\alpha(\mathbf{x}) = \frac{\hbar}{2} \sum_{\sigma,\sigma' \in \{\uparrow,\downarrow\}} G_{\sigma,\sigma'}(\mathbf{x},\tau;\mathbf{x},\tau^+)\tau^\alpha_{\sigma',\sigma} \ , \tag{5}$$

with $\tau^+ = \lim_{\eta \downarrow 0} \tau + \eta$.

d) (*10 points*) Using the Green's function in Eq. (3) evaluate the spin density. Show that this ultimately yields

$$\mathbf{s}(\mathbf{x}) = \frac{\hbar}{2} \left\{ \int \frac{d\mathbf{k}}{(2\pi)^3} \left[N_F(\epsilon_\mathbf{k} - B) - N_F(\epsilon_\mathbf{k} + B) \right] \right\} \hat{z} \equiv s(B)\hat{z} \ , \tag{6}$$

with $\hat{z}$ the unit vector in the z-direction, and $N_F(\epsilon) = [e^{\beta(\epsilon-\mu)} + 1]^{-1}$ the Fermi-Dirac distribution function. In the above we have taken the volume V large so that the sum over wave vectors $\mathbf{k}$ is written as an integral.

e) (*5 points*) Show that to first order in B we have that

$$s(B) = -\hbar B \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{\partial N_F(\epsilon_\mathbf{k})}{\partial \epsilon_\mathbf{k}} \ . \tag{7}$$

f) (*20 points*) Now we take the external field in the x-direction so that $B_x = B$, and $B_y = B_z = 0$. Determine for this situation the Green's function $G_{\sigma,\sigma'}(\mathbf{x},\tau;\mathbf{x}',\tau')$ and the spin density. Show that the result for the spin density is $\mathbf{s}(\mathbf{x}) = s(B)\hat{x}$, with $s(B)$ determined by Eq. (6), and $\hat{x}$ the unit vector in the x-direction.

g) (*5 points*) Give, without calculations, the spin density for a field in the y-direction.

h) (*10 points*) Now we consider again a field in the z direction, but make the field dependent on imaginary time and on position, so that $B_z = B(\mathbf{x},\tau)$ and $B_y = B_z = 0$. Show that the expectation value of the spin density is given by

$$\langle S(\mathbf{x},\tau) \rangle = \frac{2}{\hbar^2} \int_0^{\hbar\beta} d\tau' \int d\mathbf{x}' \langle S(\mathbf{x},\tau) S(\mathbf{x}',\tau') \rangle_0 B(\mathbf{x}',\tau') + \mathcal{O}\left(B^2\right) \ , \tag{8}$$

where $\langle \cdots \rangle_0$ denote equilibrium expectation values with magnetic field zero, and

$$S(\mathbf{x},\tau) = \frac{\hbar}{2} \sum_{\sigma,\sigma' \in \{\uparrow,\downarrow\}} \phi^*_\sigma(\mathbf{x},\tau^+)\tau^z_{\sigma,\sigma'}\phi_{\sigma'}(\mathbf{x},\tau) \ . \tag{9}$$

i) (*20 points*) Take $B(\mathbf{x},\tau) = B$ again constant. Evaluate the right-hand side of the Eq. (8). Show that this ultimately yields the same as Eq. (7).

j) (*10 points*) Evaluate Eq. (7) at $T = 0$ by carrying out the remaining integral over $\mathbf{k}$, and express the final result in terms of the total density n.