Midterm exam SFT 2011/2012

a) We have

$$S[\phi^*,\phi] = \int d\tau \int d\vec{x}\, \phi^*_\sigma(x,\tau) \left[\left(\hbar \frac{\partial}{\partial \tau} - \frac{\hbar^2 \nabla^2}{2m} - \mu\right)\delta_{\sigma\sigma'} - \sum_{\alpha\in\{x,y,z\}} B_\alpha \tau^\alpha_{\sigma\sigma'} \right] \phi_{\sigma'}(x,\tau)$$

$$\equiv \int_0^{\hbar\beta} d\tau \int dx \int_0^{\hbar\beta} d\tau' \int dx'\, \phi^*_\sigma(x,\tau) \left[-\hbar G^{-1}_{\sigma\sigma'}(x,\tau;x'\tau') \right] \phi_{\sigma'}(x'\tau')$$

where sum over $\sigma,\sigma' \in \{\uparrow,\downarrow\}$ is implied.

From this we read of:

$$-\hbar G^{-1}_{\sigma\sigma'}(x,\tau;x'\tau') = \left[\left(\hbar \frac{\partial}{\partial \tau} - \frac{\hbar^2 \nabla^2}{2m} - \mu\right)\delta_{\sigma\sigma'} - \sum_{\alpha\in\{x,y,z\}} B_\alpha \tau^\alpha_{\sigma\sigma'} \right] \delta(x-x')\,\delta(\tau-\tau')$$

[2 points]

multiplying ~~xxxxxxxx~~ with $G_{\sigma'\sigma''}(x'\tau'; x''\tau'')$,

and summing over σ', integrating over $d\tau'$ & dx'

gives:

$$\sum_{\sigma'}\left[\left(\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2\nabla^2}{2m} - \mu\right)\delta_{\sigma\sigma'} - \sum_{\alpha\in\{x,y,z\}} B_\alpha \tau^\alpha_{\sigma\sigma'}\right] G_{\sigma'\sigma''}(x\tau; x''\tau'')$$

$$= -\hbar\, \delta_{\sigma\sigma''}\, \delta(x-x'')\, \delta(\tau-\tau'') \quad [\text{3 points}]$$

Γ_1

For more details, see: UQF §7.2.3 and exercise 0.8

↳ Ultracold Quantum Fields

Choose $B_\alpha = B$ if $\alpha = z$, zero otherwise.

then (relabel $\sigma' \leftrightarrow \sigma''$; $\tau'' \to \tau'$; $x'' \to x'$)

$$\sum_{\sigma''}\left[\left(\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2\nabla^2}{2m} - \mu\right)\delta_{\sigma\sigma''} - B\,\tau^z_{\sigma\sigma''}\right] G_{\sigma''\sigma'}(x\tau; x'\tau')$$

$$= -\hbar\, \delta_{\sigma\sigma'}\, \delta(x-x')\, \delta(\tau-\tau')$$

$$\left[\left(\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2\nabla^2}{2m} - \mu\right) - \sigma B\right] G_{\sigma\sigma'}(x,\tau, x'\tau')$$

$$= -\hbar\, \delta_{\sigma\sigma'}\, \delta(x-x')\, \delta(\tau-\tau')$$

$\hbar\frac{\partial}{\partial\tau} \to -i\hbar\omega_n \qquad +\varepsilon_k$

$$\left[\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2\nabla^2}{2m} - \mu - \sigma B\right] G_{\sigma\sigma'}(x\tau, x'\tau')$$

$$= \frac{\delta_{\sigma\sigma'}}{\hbar\beta V}\sum_{k,n} -\hbar\left(\frac{-i\hbar\omega_n + \varepsilon_k - \mu - \sigma B}{-i\hbar\omega_n + \varepsilon_k - \mu - \sigma B}\right) e^{ik\cdot(x-x') - i\omega_n(\tau-\tau')}$$

[5 points]

$$= -\hbar\frac{\delta_{\sigma\sigma'}}{\hbar\beta V}\sum_{k,n} e^{ik\cdot\vec{x}-\vec{x}' - i\omega_n(\tau-\tau')}$$

$$= -\hbar\,\delta_{\sigma\sigma'}\frac{1}{V}\sum_{k} e^{ik(x-x')}\left(\frac{1}{\hbar\beta}\sum_n e^{-i\omega_n(\tau-\tau')}\right)$$

$$= -\hbar\,\delta_{\sigma\sigma'}\,\delta(x-x')\qquad \delta(\tau-\tau')$$

[5 points]

c) $$G_{\sigma\sigma'}(x,\tau, x'\tau') = -\left\langle T\left[\hat{\psi}_\sigma(x\tau)\,\hat{\psi}^\dagger_{\sigma'}(x'\tau')\right]\right\rangle$$

[3 points]

So: $$G_{\sigma\sigma'}(x\tau, x\tau^+) = \left\langle \hat{\psi}^\dagger_{\sigma'}(\vec{x})\,\psi_\sigma(\vec{x})\right\rangle$$

then $$S_\alpha = \frac{\hbar}{2}\sum_{\sigma\sigma'}\left\langle \hat{\psi}^\dagger_\sigma(\vec{x})\,\tau^\alpha_{\sigma\sigma'}\,\psi_{\sigma'}(\vec{x})\right\rangle$$

$$= \frac{\hbar}{2}\sum_{\sigma\sigma'}\tau^\alpha_{\sigma\sigma'}\left\langle \hat{\psi}^\dagger_\sigma(x)\,\psi_{\sigma'}(x)\right\rangle$$

$$= \frac{\hbar}{2}\sum_{\sigma\sigma'}\tau^\alpha_{\sigma\sigma'}\,G_{\sigma'\sigma}(x\tau; x\tau^+)$$

[2 points]

d) Using (3) and (5):

$\overset{\alpha}{S}(\vec{x}) = 0$ if $\alpha = x$ or y.

$$S^z(\vec{x}) = \frac{\hbar}{2} \frac{1}{\hbar\beta V} \sum_{k,n} e^{i\omega_n \eta} \left[\frac{1}{i\omega_n - (\varepsilon_k - \mu - B)/\hbar} - \frac{1}{i\omega_n - (\varepsilon_k - \mu + B)/\hbar} \right]$$

Carry out $\sum_n$ using 7.31

$$= \frac{\hbar}{2} \frac{1}{V} \sum_{k} \left[N_F(\varepsilon_k - B) - N_F(\varepsilon_k + B) \right] \quad (8 \text{ points})$$

$$= \frac{\hbar}{2} \int \frac{dk}{(2\pi)^3} \left[N_F(\varepsilon_k - B) - N_F(\varepsilon_k + B) \right] \quad [2 \text{ points}]$$

e) see exercise 0.1 b)

$$N_F(\varepsilon_k + E) \cong N_F(\varepsilon_k) + N_F'(\varepsilon_k) E + \mathcal{O}(E^2)$$

$$N'(x) \equiv \frac{\partial N}{\partial x}$$

rest follows.

f) Use result Γ_1 with $B^\alpha = B$ if $\alpha = z$, zero otherwise

$$\begin{pmatrix} \left(\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2\nabla^2}{2m} - \mu\right) & -B \\ -B & \hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2\nabla^2}{2m} - \mu \end{pmatrix} \underline{G}(x,\tau; x'\tau')$$

see exercise 0.8 for G's with spin-$\frac{1}{2}$

2 x 2 matrix in spin space

$$= -\hbar \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \delta(x - x'') \, \delta(\tau - \tau')$$

Fourier transform:

$$\begin{pmatrix} (-i\hbar\omega_n + \varepsilon_k - \mu) & -B \\ -B & (-i\hbar\omega_n + \varepsilon_k - \mu) \end{pmatrix} \underline{G}(k, i\omega_n) = -\hbar \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \underline{G}(k, i\omega_n) = \frac{-\hbar}{\left[(-i\hbar\omega_n + \varepsilon_k - \mu)^2 - B^2\right]} \times \begin{pmatrix} (-i\hbar\omega_n + \varepsilon_k - \mu) & +B \\ +B & (-i\hbar\omega_n + \varepsilon_k - \mu) \end{pmatrix}$$

~~with~~

[10 points]

(6)

Use:

$$S^{\alpha}(\vec{x}) = \frac{\hbar}{2} \frac{1}{\hbar\beta V} \sum_{k, i\omega_n} \mathrm{Tr}\left[\underline{\underline{G}}(k, i\omega_n)\, \tau^{\alpha} \right]$$

$= ~~exit~~$ 0 if $\alpha \neq x$

then:

$$S^{x}(\vec{x}) = \frac{\hbar}{2} \frac{1}{\hbar\beta V} \sum_{k, i\omega_n}$$

$$\times \, \mathrm{Trace}\left[\frac{-\hbar}{(-i\hbar\omega_n + \varepsilon_k - \mu)^2 - B^2} \begin{pmatrix} (-i\hbar\omega_n + \varepsilon_k - \mu) & +B \\ +B & (-i\hbar\omega_n + \varepsilon_k - \mu) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right]$$

$$= \frac{\hbar}{2} \frac{1}{\hbar\beta V} \sum_{k, i\omega_n} \frac{-\hbar}{(-i\hbar\omega_n + \varepsilon_k - \mu)^2 - B^2} \underbrace{\mathrm{Tr} \begin{pmatrix} B & -i\hbar\omega_n + \varepsilon_k - \mu \\ -i\hbar\omega_n + \varepsilon_k - \mu & B \end{pmatrix}}_{2B}$$

$$= \frac{\hbar}{2} \frac{1}{\hbar\beta V} \sum_{k, n} \frac{-\hbar}{(-i\hbar\omega_n + \varepsilon_k - \mu)^2 - B^2} \times 2B$$

split fractions.

$$= -\frac{\hbar^2}{2} \frac{1}{\hbar\beta V} \sum_{k, n} \left[\frac{1}{i\hbar\omega_n + \varepsilon_k - \mu - B} - \frac{1}{i\hbar\omega_n + \varepsilon_k - \mu + B} \right]$$

rest is same as part d)

[10 points]

g) $\vec{S}(\vec{x}) = S(B)\,\hat{y}$ because of rotational symmetry

h) see exercise 0.11

$$\langle S(\vec{x},\tau)\rangle = \frac{\int d[q^*]\,d[q]\; e^{-\frac{1}{\hbar}S_0[q^*,q] + \frac{2}{\hbar^2}\int d\tau' \int dx'\, B(x',\tau')\,S(x'\tau')}\; S(x\tau)}{\int d[q^*]\,d[q]\; e^{-\frac{1}{\hbar}S_0[q,q^*] + \frac{1}{\hbar}\int d\tau'\int dx'\, B(x'\tau')\,S(x'\tau')}}$$

$$\simeq \frac{\int d[q^*]\,d[q]\; e^{-\frac{1}{\hbar}S_0[q^*,q]}\left(1 + \frac{2}{\hbar^2}\int d\tau'\int dx'\, B(x'\tau')\,S(x'\tau')\right)S(x,\tau) + \dots}{\int d[q^*]\,d[q]\; e^{-\frac{1}{\hbar}S_0[q^*,q]}\left(1 + \frac{2}{\hbar^2}\int d\tau'\int dx'\, B(x'\tau')\,S(x'\tau') + \dots\right)}$$

$\to 0$ follows from e.g. Eq (3) with $B=0$.

$$= \frac{Z_0\,\langle S(x,\tau)\rangle_0 + \frac{2}{\hbar^2} Z_0 \int d\tau' \int dx' \left(\langle S(x,\tau)\,S(x'\tau')\rangle_0\, B(x'\tau')\right)}{Z_0 + Z_0\,\frac{2}{\hbar}\langle S\rangle_0} \quad \leftarrow [6\text{ points}] \quad \leftarrow [4\text{ points}]$$

$\langle S\rangle_0 \to 0$

$$= \frac{2}{\hbar^2}\int d\tau' \int dx'\, \langle S(x\tau)\,S(x'\tau')\rangle_0\, B(x'\tau')$$

i) see exercise 0.11

$$\cancel{\int_0^{\hbar\beta} d\tau' \int dx'} \frac{4}{\hbar^2}\langle S(x,\tau)\, S(x'\tau')\rangle_0$$

$$= \left\langle \left(\phi_\uparrow^*(x\tau)\,\phi_\uparrow(x\tau) - \phi_\downarrow^*(x\tau)\,\phi_\downarrow(x\tau)\right)\left(\phi_\uparrow^*(x\tau')\,\phi_\uparrow(x\tau') - \phi_\downarrow^*(x'\tau')\,\phi_\downarrow(x'\tau')\right)\right\rangle_0$$

Wick's theorem [5 points]

$$\overset{!}{=} \quad 2 \times \langle \phi_\uparrow^*(x\tau)\,\phi_\uparrow(x\tau)\,\phi_\uparrow^*(x'\tau')\,\phi_\uparrow(x'\tau')\rangle + \text{other}$$

because the $\downarrow$-term is the same and $\langle \phi_\uparrow^* \phi_\downarrow\rangle \to 0$ [5 points]

$$= -2\; G_{\uparrow\uparrow}(x\tau; x'\tau')\; G_{\uparrow\uparrow}(x'\tau', x\tau)$$

So: $\displaystyle \frac{2B}{\hbar^2}\int_0^{\hbar\beta} d\tau' \int dx' \langle S(x\tau)\, S(x'\tau')\rangle_0$

$$= -B \int d\tau' \int dx' \; G_{\uparrow\uparrow}(x\tau; x'\tau')\, G_{\uparrow\uparrow}(x',\tau'; x\tau)$$

$$= -B \int d\tau' \int dx' \frac{1}{\hbar\beta V}\sum_{k, i\omega_n} \frac{1}{(\hbar\beta V)} \sum_{k', i\omega_n'} G_{\uparrow\uparrow}(k, i\omega_n)\, G_{\uparrow\uparrow}(k', i\omega_n')\; e^{ik(x-x') - ik'(x-x') - i\omega_n(\tau-\tau') + i\omega_{n'}(\tau'-\tau)}$$

$$\overset{!}{=} \frac{-\hbar}{-i\hbar\omega_n + \epsilon_k - \mu}$$

$d\tau' \to \hbar\beta\, \delta_{n,n'}$ $\quad \int dx' \to V\, \delta_{k,k'}$ [$\longleftarrow$ 5 points up to here]

$$= -B \frac{1}{\hbar\beta V}\sum_{k, i\omega_n}\left(\frac{-\hbar}{-i\hbar\omega_n + \epsilon_k - \mu}\right)^2 = +B \frac{1}{\hbar\beta V}\sum_{k}\frac{\partial}{\partial \epsilon_k}\sum_n \left(\frac{+\hbar^2}{-i\hbar\omega_n + \epsilon_k - \mu}\right)$$

$= \dots$

$$\ldots = \mp B\hbar \frac{1}{V}\sum_{k} \frac{\partial}{\partial \varepsilon_k} \left(\frac{1}{\hbar\beta} \sum_n \frac{\partial}{\partial \varepsilon_k} \frac{1}{i\omega_n - (\varepsilon_k - \mu)/\hbar} \right)$$

$$= -\hbar\beta \int \frac{dk}{(2\pi)^3} \frac{\partial}{\partial \varepsilon_k} \qquad N_F(\varepsilon_k - \mu)$$

You could also use exercise 0.6 a) and b)

and then take $k \to 0$, $i\omega_n \to 0$

in (3).

[5 points end result]

j) Exercise 0.1 e

$$\int \frac{dk}{(2\pi)^3} \frac{\partial N_F(\varepsilon_k)}{\partial \varepsilon_k} = \frac{4\pi}{(2\pi)^3} \int dk\, k^2 \frac{\partial N_F(\varepsilon_k)}{\partial \varepsilon_k} \qquad \left|\; k^2 = \frac{2m\varepsilon}{\hbar^2} \quad \frac{d\varepsilon}{dk} = \frac{\hbar^2 k}{m} = \frac{\hbar^2}{m}\sqrt{\frac{2m\varepsilon}{\hbar^2}} \right.$$

$$= \frac{4\pi}{(2\pi)^3} \int d\varepsilon \sqrt{\varepsilon} \frac{\partial N}{\partial \varepsilon} \times \left(\frac{2m}{\hbar^2} \frac{m}{\hbar^2} \sqrt{\frac{\hbar^2}{2m}} \right)$$

$$= \frac{1}{2\pi^2 \sqrt{2}} \frac{m^{3/2}}{\hbar^3} \int d\varepsilon \sqrt{\varepsilon} \frac{\partial N}{\partial \varepsilon}$$

$$= \frac{1}{2\pi^2\sqrt{2}} \frac{m^{3/2}}{\hbar^3} \int d\varepsilon \sqrt{\varepsilon} \; -\delta(\varepsilon - \varepsilon_F) = \frac{-1}{2\pi^2\sqrt{2}} \frac{m^{3/2}}{\hbar^3} \sqrt{\varepsilon_F}$$

where ε_F is determined from:

$$n = 2 \int_0^{\varepsilon_F} d\varepsilon \; \frac{1}{2\pi^2\sqrt{2}} \frac{m^{3/2}}{\hbar^3} \sqrt{\varepsilon}$$

$$= 2 \, \frac{1}{\pi^2\sqrt{2}} \frac{m^{3/2}}{\hbar^3} \frac{2}{3} \varepsilon_F^{3/2}$$

in terms of total density n

for spin-$\frac{1}{2}$