INSTITUTE FOR THEORETICAL PHYSICS
UTRECHT UNIVERSITY

Exam Statistical Field Theory (NS-TP402M)

Tuesday, January 29, 2013, 15:00-18:00

1. *Use a separate sheet for every exercise.*
2. *Write your name and initials on all sheets, on the first sheet also your student ID number.*
3. *Write clearly, unreadable work cannot be corrected.*
4. *You may use your notes, solutions to exercises, and the book by Stoof et al.*
5. *Distribute your time evenly over the exam, don't spend an enormous amount of time on correcting minus signs, factors of two and/or π, etc. In case you suspect that you have made a calculational error but don't have enough time to correct it, point it out in words.*
6. *Give the motivation, explanation, and calculations leading up to each answer and/or solution.*

Exercise 1: Density wave in a one-dimensional Fermi gas (*70 points*)

In this exercise, we consider a one-dimensional homogeneous Fermi gas. We do not consider the spin of the particles. The potential for pair-wise interactions between the fermions is denoted by $V(x-x')$, and their mass with m. Upon lowering the temperature, this system can undergo a phase transition to a density wave, with the density $\rho(x,\tau)$ given by

$$\langle \rho(x,\tau)\rangle \equiv \langle \phi^*(x,\tau)\phi(x,\tau)\rangle = \rho_0 + A\cos(qx+\theta)\ .$$

Here, A, q, and θ are real numbers, with A the amplitude of the density wave, θ the phase, and q its wave number. Furthermore, ρ_0 is the average density.

To describe the phase transition, we start out with the grand-canonical partition function given by

$$Z = \int d[\phi^*]\, d[\phi] \exp\{-S[\phi^*,\phi]/\hbar\}\ , \tag{1}$$

where the action is given by

$$\begin{aligned} S[\phi^*,\phi] &= \int_0^{\hbar\beta} d\tau \int dx \phi^*(x,\tau)\left[\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - \mu\right]\phi(x,\tau) \\ &+ \int_0^{\hbar\beta} d\tau \int dx \int dx' \phi^*(x,\tau)\phi^*(x',\tau)\frac{V(x-x')}{2}\phi(x',\tau)\phi(x,\tau)\ , \end{aligned} \tag{2}$$

with μ the chemical potential, $\hbar$ Planck's constant, and $\beta = 1/k_BT$ the inverse thermal energy.

a) (*5 points*) For which type of interaction do you expect this phase transition? Argue that the system, as described by the above action, does not exhibit a phase transition if we approximate the interaction by means of $V(x-x') = V_0\delta(x-x')$.

b) (*7 points*) Decouple the interaction with a Hubbard-Stratonovich transformation to the density $\rho(x,\tau)$ and show that after the transformation we have that

$$Z = \int d[\rho]\, d[\phi^*]\, d[\phi] \exp\{-S[\phi^*,\phi,\rho]/\hbar\} \ ,$$

with the action

$$S[\phi^*,\phi,\rho] = \int_0^{\hbar\beta} d\tau \int dx \phi^*(x,\tau)\left[\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - \mu\right]\phi(x,\tau)$$
$$- \frac{1}{2}\int_0^{\hbar\beta} d\tau \int dx \int dx' \left\{\rho(x,\tau)V(x-x')\rho(x',\tau) - 2\rho(x',\tau)V(x-x')\phi^*(x,\tau)\phi(x,\tau)\right\} \ . \qquad (3)$$

c) (*8 points*) Perform the functional integration over the fields ϕ^* and ϕ, so that the expression for Z is written as

$$Z = \int d[\rho] \exp\{-S_{\text{eff}}[\rho]/\hbar\} \ ,$$

and give the exact but formal expression for the effective action $S_{\text{eff}}[\rho]$.

d) (*15 points*) Write $\rho(x,\tau) = \rho_0 + \delta\rho(x,\tau)$, and give the equation for ρ_0 (which is independent of x and τ) by demanding that terms linear in the effective action in $\delta\rho(x,\tau)$ vanish. Calculate the effective action up to quadratic order in the density-fluctuation field $\delta\rho(x,\tau)$ and show that (up to irrelevant constants)

$$\begin{aligned} S_{\text{eff}}[\delta\rho] &= -\frac{1}{2}\int_0^{\hbar\beta} d\tau \int dx \int_0^{\hbar\beta} d\tau' \int dx' \delta\rho(x,\tau)\Big[V(x-x')\delta(\tau-\tau') \\ &-\frac{1}{\hbar}\int dx'' \int dx''' V(x''-x)G_0(x''-x''';\tau-\tau')G_0(x'''-x'';\tau'-\tau)V(x'''-x')\Big]\delta\rho(x',\tau') \ , \end{aligned} \qquad (4)$$

where $G_0(x-x';\tau-\tau')$ is the single-particle Green's function in the Hartree approximation. NB: From now on, ignore the Hartree mean-field shift.

e) (*15 points*) To describe the phase transition we consider static, i.e., time-independent, configurations of spatial fluctuations in the density, which we write as

$$\delta\rho(x) = \sum_k \delta\rho(k) e^{ikx} \ .$$

Show that for these configurations the above effective action becomes

$$S_{\text{eff}}[\delta\rho] = -\frac{1}{2}\hbar\beta L \sum_k \left[\tilde{V}(k) - \tilde{V}(k)\Pi(k)\tilde{V}(k)\right]\delta\rho^*(k)\delta\rho(k) \ .$$

with

$$\Pi(k) = \frac{1}{L}\sum_q \frac{N_F(\epsilon_{k+q}) - N_F(\epsilon_q)}{\epsilon_{k+q} - \epsilon_q} \ ,$$

where L is the length of the system, $\epsilon_k = \hbar^2 k^2/(2m)$, and $N_F(x) = [e^{\beta(x-\mu)}+1]^{-1}$ is the Fermi-Dirac distribution function. Furthermore, $\tilde{V}(k)$ is the Fourier transform of the interaction potential $V(x-x')$.

The function $\Pi(k)$ is sharply peaked for $k = 2k_F$. In view of this result, we keep only the $k = 2k_F$ component of the effective action, so that we take

$$S_{\text{eff}}[\delta\rho] = -\frac{1}{2}\hbar\beta L\left[\tilde{V}(2k_F) - \tilde{V}(2k_F)\Pi(2k_F)\tilde{V}(2k_F)\right]\delta\rho^*(2k_F)\delta\rho(2k_F) \ .$$

f) (*10 points*) We define a Landau free energy density $f_L(|\delta\rho(2k_F)|)$ according to $S_{\text{eff}}[\delta\rho] = \hbar\beta L f_L(|\delta\rho(2k_F)|)$. Argue that this Landau free energy describes the phase transition to the density wave, and give its wave vector q. Give the equation that determines the critical temperature of the transition without evaluating it.

g) (*10 points*) Show that the single-particle spectrum becomes gapped below the critical temperature, and that the gap is proportional to the amplitude of the density wave.

Exercise 2: mean-field theory for a planar magnet (*30 points*)

Consider a gas of spin one-half fermionic atoms with mass m, in a box of volume V. Consider the situation that these undergo a ferromagnetic phase transition, with spin direction in the $x-y$-plane (a so-called planar magnet). This transition is signalled by a nonzero expectation value $\langle\hat{\psi}^\dagger_\uparrow(\mathbf{x})\hat{\psi}_\downarrow(\mathbf{x})\rangle \equiv \Psi$ that is independent of position, with $\hat{\psi}_\alpha(\mathbf{x})$ and $\hat{\psi}^\dagger_\alpha(\mathbf{x})$ the annihilation and creation operator for an atom with spin state $|\alpha\rangle$. Here, $\alpha \in \{\uparrow,\downarrow\}$ refers to spin projections with the z-axis as quantization axis.

The hamiltonian for the system is given by

$$\hat{H} = \int d\mathbf{x} \left\{ \sum_{\alpha\in\{\uparrow,\downarrow\}} \hat{\psi}^\dagger_\alpha(\mathbf{x}) \left[-\frac{\hbar^2\nabla^2}{2m} - \mu \right] \hat{\psi}_\alpha(\mathbf{x}) + V_0 \hat{\psi}^\dagger_\uparrow(\mathbf{x})\hat{\psi}^\dagger_\downarrow(\mathbf{x})\hat{\psi}_\downarrow(\mathbf{x})\hat{\psi}_\uparrow(\mathbf{x}) \right\} ,$$

where we approximated the interaction with a short-range potential with strength V_0.

a) (*5 points*) Express the spin density in terms of Ψ and show that it indeed describes a planar ferromagnnet.

b) (*10 points*) Give the hamiltonian in the mean-field approximation, where

$$\hat{\psi}^\dagger_\uparrow(\mathbf{x})\hat{\psi}^\dagger_\downarrow(\mathbf{x})\hat{\psi}_\downarrow(\mathbf{x})\hat{\psi}_\uparrow(\mathbf{x}) \simeq -\hat{\psi}^\dagger_\uparrow(\mathbf{x})\langle\hat{\psi}^\dagger_\downarrow(\mathbf{x})\hat{\psi}_\uparrow(\mathbf{x})\rangle\hat{\psi}_\downarrow(\mathbf{x}) - \hat{\psi}^\dagger_\downarrow(\mathbf{x})\langle\hat{\psi}^\dagger_\uparrow(\mathbf{x})\hat{\psi}_\downarrow(\mathbf{x})\rangle\hat{\psi}_\uparrow(\mathbf{x}) + |\Psi|^2 .$$

From now on, take Ψ real. Show that after Fourier transformation we have that

$$\hat{H} = \sum_{\mathbf{k};\alpha,\alpha'\in\{\uparrow,\downarrow\}} \hat{c}^\dagger_{\mathbf{k},\alpha} \left[(\epsilon_{\mathbf{k}} - \mu)\, \delta_{\alpha,\alpha'} - V_0 \Psi \tau^x_{\alpha,\alpha'} \right] \hat{c}_{\mathbf{k},\alpha'} + V_0 V \Psi^2 ,$$

with τ^x a Pauli matrix.

c) (*15 points*) Derive the mean-field equation for Ψ.