Note: these solutions have been made by a student

SFT Mid Term 2012-2013

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a

$[\hat{\psi}(\mathbf{x}), \hat{\psi}^\dagger(\mathbf{x}')] = \delta(\mathbf{x}-\mathbf{x}')$ other commutators are zero [See also (6.28) and note that we are looking at spinless bosons.]

b

Invert the given expression for the field operator expressed in the basis of the single particle eigenstates: $\hat{c}_n = \int d\mathbf{x}\psi(\mathbf{x})\chi_n^*(\mathbf{x}) = \sum_{n'} \hat{c}_{n'} \int \chi_{n'}(\mathbf{x})\chi_n^*(\mathbf{x}) = \sum_{n'} \hat{c}_{n'}\delta_{n',n} = \hat{c}_n$. Where we used the orthonormality of the eigenstates. Likewise, taking the hermitean conjugate of the given expression, gives you $\hat{c}_n^\dagger = \int d\mathbf{x}\psi^\dagger(\mathbf{x})\chi_n(\mathbf{x})$
Then

$$[\hat{c}_n, \hat{c}_{n'}^\dagger] = [\int d\mathbf{x}\hat{\psi}(\mathbf{x})\chi_n^*(\mathbf{x}), \int d\mathbf{x}'\hat{\psi}^\dagger(\mathbf{x}')\chi_{n'}(\mathbf{x}')] = \int d\mathbf{x} \int d\mathbf{x}'\chi_{n'}(\mathbf{x}')\chi_n^*(\mathbf{x})[\hat{\psi}(\mathbf{x}), \hat{\psi}^\dagger(\mathbf{x}')]$$
$$= \int d\mathbf{x} \int d\mathbf{x}'\chi_{n'}(\mathbf{x}')\chi_n^*(\mathbf{x})\delta(\mathbf{x}-\mathbf{x}') = \int d\mathbf{x}\chi_{n'}(\mathbf{x})\chi_n^*(\mathbf{x}) = \delta_{n',n}$$

Where we used the linearity of the commutator.

c

$$\hat{H} = \int d\mathbf{x}\hat{\psi}^\dagger(\mathbf{x})\left[-\frac{\hbar^2\nabla^2}{2m} + V^{ex}(\mathbf{x})\right]\hat{\psi}(\mathbf{x}) = \sum_{n,n'}\int d\mathbf{x}\hat{c}_n^\dagger\chi_n(\mathbf{x})\left[-\frac{\hbar^2\nabla^2}{2m} + V^{ex}(\mathbf{x})\right]\hat{c}_{n'}\chi_{n'}(\mathbf{x}) = \sum_{n,n'}\int d\mathbf{x}\hat{c}_n^\dagger\chi_n(\mathbf{x})\epsilon_{n'}\hat{c}_{n'}\chi_{n'}(\mathbf{x})$$
$$= \sum_{n,n'}\delta_{n',n}\hat{c}_n^\dagger\epsilon_{n'}\hat{c}_{n'} = \sum_n \epsilon_n\hat{c}_n^\dagger\hat{c}_n$$

Where ϵ_n is the energy of the single particle eigenstate χ_n.

d

From the two expression for the action it follows that $-\hbar G^{-1}(\mathbf{x},\tau;\mathbf{x}',\tau') = \left[\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2\nabla^2}{2m} + V^{ex}(\mathbf{x}) - \mu\right]\delta(\mathbf{x}-\mathbf{x}')\delta(\tau-\tau')$. Multiplying both sides with $G(\mathbf{x}'',\tau'';\mathbf{x},\tau)$ and integrating over $\mathbf{x}$ and τ gives you

$$-\hbar\delta(\mathbf{x}''-\mathbf{x}')\delta(\tau''-\tau') = -\hbar\int \mathbf{x}\int \tau G^{-1}(\mathbf{x},\tau;\mathbf{x}',\tau')G(\mathbf{x}'',\tau'';\mathbf{x},\tau)$$
$$= \int \mathbf{x}\int \tau\left[\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2\nabla^2}{2m} + V^{ex}(\mathbf{x}) - \mu\right]\delta(\mathbf{x}-\mathbf{x}')\delta(\tau-\tau')G(\mathbf{x}'',\tau'';\mathbf{x},\tau)$$
$$= \left[\hbar\frac{\partial}{\partial\tau'} - \frac{\hbar^2\nabla^2}{2m} + V^{ex}(\mathbf{x}') - \mu\right]G(\mathbf{x}'',\tau'';\mathbf{x}',\tau')$$

As is done on page 144 of Stoof.

$$\left[\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2\nabla^2}{2m} + V^{ex}(\mathbf{x}) - \mu\right]\frac{1}{\hbar\beta}\sum_{\mathbf{n},n}\frac{-\hbar}{-i\hbar\omega_n + \epsilon_{\mathbf{n}} - \mu}\chi_{\mathbf{n}}(x)\chi_{\mathbf{n}}^*(x')e^{-i\omega_n(\tau-\tau')}$$
$$= \frac{1}{\hbar\beta}\sum_{\mathbf{n},n}\frac{-\hbar(-i\hbar\omega_n + \epsilon_{\mathbf{n}} - \mu)}{-i\hbar\omega_n + \epsilon_{\mathbf{n}} - \mu}\chi_{\mathbf{n}}(x)\chi_{\mathbf{n}}^*(x')e^{-i\omega_n(\tau-\tau')}$$
$$= -\hbar\sum_{\mathbf{n}}\chi_{\mathbf{n}}(x)\chi_{\mathbf{n}}^*(x')\sum_{n'}\frac{e^{-in'(\tau-\tau')}}{2\pi} = -\hbar\delta(\mathbf{x}-\mathbf{x}')\delta(\tau-\tau')$$

Where we changed the coordinates from n to $n' = \frac{2\pi n}{\hbar\beta}$

e

$\chi_{\mathbf{n}}(\mathbf{x}) = \frac{e^{i\mathbf{kx}}}{\sqrt{V}}$, $\epsilon_k = \frac{\hbar^2 k^2}{2m}$ (follows from time-independent Schrödinger equation)
So $G(\mathbf{x},\tau;\mathbf{x}',\tau') = \frac{1}{\hbar\beta V}\sum_{\mathbf{k},n}\frac{-\hbar}{-i\hbar\omega_n+\epsilon_{\mathbf{k}}-\mu}e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')}e^{-i\omega_n(\tau-\tau')}$. (page 145)

f

$n_0 = \langle\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}(\mathbf{x})\rangle$ is a commuting number.

$$\begin{aligned}
[\delta\hat{n}(\mathbf{x}),\delta\hat{n}(\mathbf{x}')] &= [\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}(\mathbf{x})-n_0,\hat{\psi}^\dagger(\mathbf{x}')\hat{\psi}(\mathbf{x}')-n_0] = [\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}(\mathbf{x}),\hat{\psi}^\dagger(\mathbf{x}')\hat{\psi}(\mathbf{x}')]-0-0+0\\
&= \hat{\psi}^\dagger(\mathbf{x})\hat{\psi}(\mathbf{x})\hat{\psi}^\dagger(\mathbf{x}')\hat{\psi}(\mathbf{x}') - \hat{\psi}^\dagger(\mathbf{x}')\hat{\psi}(\mathbf{x}')\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}(\mathbf{x})\\
&= \hat{\psi}^\dagger(\mathbf{x})\left[\delta(\mathbf{x}-\mathbf{x}')-\hat{\psi}^\dagger(\mathbf{x}')\hat{\psi}(\mathbf{x})\right]\hat{\psi}(\mathbf{x}') - \hat{\psi}^\dagger(\mathbf{x}')\left[\delta(\mathbf{x}-\mathbf{x}')-\hat{\psi}(\mathbf{x}')\hat{\psi}^\dagger(\mathbf{x})\right]\hat{\psi}(\mathbf{x})\\
&= (\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}(\mathbf{x}')-\hat{\psi}^\dagger(\mathbf{x}')\hat{\psi}(\mathbf{x}))\delta(\mathbf{x}-\mathbf{x}') + \hat{\psi}^\dagger(\mathbf{x}')\hat{\psi}(\mathbf{x}')\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}(\mathbf{x})(1-1)\\
&= \delta(\mathbf{x}-\mathbf{x}')(\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}(\mathbf{x}')-\hat{\psi}^\dagger(\mathbf{x}')\hat{\psi}(\mathbf{x}))
\end{aligned}$$

Where we repeatedly used the commutation relations of the field operators.

g

Note that "in equilibrium" means practically that we can use the techniques we have learned thus far in the course SFT.
$\langle\delta\hat{n}(\mathbf{x})\rangle = \langle\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}(\mathbf{x})-n_0\rangle = \langle\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}(\mathbf{x})-\langle\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}(\mathbf{x})\rangle\rangle = \langle\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}(\mathbf{x})\rangle - \langle\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}(\mathbf{x})\rangle = 0$

$$\begin{aligned}
\langle\hat{n}(\mathbf{x},\mathbf{x}')\rangle &= \langle\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}(\mathbf{x}')\rangle = \lim_{\eta\downarrow 0}\langle T[\hat{\psi}(\mathbf{x}',\tau)\hat{\psi}^\dagger(\mathbf{x},\tau^+)]\rangle = \lim_{\eta\downarrow 0} -G(\mathbf{x},\tau;\mathbf{x},\tau^+) = \lim_{\eta\downarrow 0}\frac{1}{\hbar\beta V}\sum_{\mathbf{k},n}\frac{\hbar}{-i\hbar\omega_n+\epsilon_{\mathbf{k}}-\mu}e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')}e^{i\omega_n\eta}\\
&= \frac{1}{V}\sum_{\mathbf{k}}\frac{1}{e^{\beta(\epsilon-\mu)}-1}e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')} = \frac{1}{(2\pi)^3}\int d\mathbf{k}\frac{1}{e^{\beta(\epsilon-\mu)}-1}e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')}\\
&= \frac{1}{(2\pi)^3}\int_0^\infty dk\int_0^{2\pi}d\phi\int_0^\pi d\theta k^2\sin(\theta)N_B(\epsilon_k)e^{i\cos(\theta)|\mathbf{k}||\mathbf{x}-\mathbf{x}'|}\\
&= \frac{\text{-1}}{i(2\pi)^2|x-x'|}\int_0^\infty dkkN_B(\epsilon_k)(e^{-i\mathbf{k}|\mathbf{x}-\mathbf{x}'|}-e^{i\mathbf{k}|\mathbf{x}-\mathbf{x}'|})\\
&= \frac{1}{2\pi^2|x-x'|}\int_0^\infty dkkN_B(\epsilon_k)\sin(k|x-x'|)
\end{aligned}$$

Where we used (7.31) and afterwards similar strategies as in exercise 0.1.

h

$$\begin{aligned}
\langle\delta\hat{n}(\mathbf{x},\tau)\delta\hat{n}(\mathbf{x}',\tau')\rangle &= \langle(\phi^*(\mathbf{x},\tau)\phi(\mathbf{x},\tau)-n_0)(\phi^*(\mathbf{x}',\tau')\phi(\mathbf{x}',\tau')-n_0)\rangle = \langle\phi^*(\mathbf{x},\tau)\phi(\mathbf{x},\tau)\phi^*(\mathbf{x}',\tau')\phi(\mathbf{x}',\tau')\rangle\\
&- n_0\langle\phi^*(\mathbf{x},\tau)\phi(\mathbf{x},\tau)\rangle - n_0\langle\phi^*(\mathbf{x}',\tau')\phi(\mathbf{x}',\tau')\rangle + n_0^2 = \langle\phi^*(\mathbf{x},\tau)\phi(\mathbf{x},\tau)\phi^*(\mathbf{x}',\tau')\phi(\mathbf{x}',\tau')\rangle - n_0^2\\
&= \langle\phi^*(\mathbf{x},\tau)\phi(\mathbf{x}',\tau')\rangle\langle\phi^*(\mathbf{x}',\tau')\phi(\mathbf{x},\tau)\rangle + \langle\phi^*(\mathbf{x},\tau)\phi(\mathbf{x},\tau)\rangle\langle\phi^*(\mathbf{x}',\tau')\phi(\mathbf{x}',\tau')\rangle - n_0^2\\
&= (\text{-1})^2G(\mathbf{x},\tau;\mathbf{x}',\tau')G(\mathbf{x}',\tau';\mathbf{x},\tau) + (\text{1-1})n_0^2 = G(\mathbf{x},\tau;\mathbf{x}',\tau')G(\mathbf{x}',\tau';\mathbf{x},\tau)
\end{aligned}$$

Where we used wick's theorem (7.72).

i

$$\begin{aligned}
\Pi(\mathbf{k},i\omega_n) &= \int d(\mathbf{x}-\mathbf{x}')\int_0^{\hbar\beta}d(\tau-\tau')\frac{1}{(\hbar\beta)^2}\sum_{\mathbf{k}'',n'',\mathbf{k}',n'}\frac{e^{i(x-x')(k''-k')-i(\omega_{n''}-\omega_{n'})(\tau-\tau')}}{(i\omega_{n''}-(\epsilon_{\mathbf{k}''}-\mu)/\hbar)(i\omega_{n'}-(\epsilon_{\mathbf{k}'}-\mu)/\hbar)}e^{-i\mathbf{k}(\mathbf{x}-\mathbf{x}')+i\omega_n(\tau-\tau')}\\
&= \frac{1}{(\hbar\beta)V}\sum_{\mathbf{k}'',n'',\mathbf{k}',n'}\frac{\delta(\mathbf{k}-\mathbf{k}''+\mathbf{k}')\delta(\omega_n-\omega_{n''}+\omega_{n'})}{(i\omega_{n''}-(\epsilon_{\mathbf{k}''}-\mu)/\hbar)(i\omega_{n'}-(\epsilon_{\mathbf{k}'}-\mu)/\hbar)}\\
&= \frac{1}{(\hbar\beta)V}\sum_{\mathbf{k}',n'}\frac{1}{(i(\omega_n+\omega_{n'})-(\epsilon_{\mathbf{k}}+\epsilon_{\mathbf{k}'}-\mu)/\hbar)(i\omega_{n'}-(\epsilon_{\mathbf{k}'}-\mu)/\hbar)}
\end{aligned}$$

$\frac{1}{AB} = \frac{1}{A-B}(\frac{1}{A} - \frac{1}{B})$

$$
\begin{aligned}
\Pi(\mathbf{k}, i\omega_n) &= \frac{1}{(\hbar\beta)V} \sum_{\mathbf{k}', n'} \frac{1}{i\omega_n - (\epsilon_{\mathbf{k}+\mathbf{k}'} - \epsilon_{\mathbf{k}'})/\hbar} \left[\frac{1}{i(\omega_n + \omega_{n'}) - (\epsilon_{\mathbf{k}+\mathbf{k}'} - \mu)/\hbar} - \frac{1}{i\omega_{n'} - (\epsilon_{\mathbf{k}'} - \mu)/\hbar} \right] \\
&= \frac{1}{(2\pi)^3} \int \mathbf{k}' \frac{N_B(\epsilon_{\mathbf{k}+\mathbf{k}'}) - N_B(\epsilon_{\mathbf{k}'})}{i\omega_n - (\epsilon_{\mathbf{k}+\mathbf{k}'} - \epsilon_{\mathbf{k}'})/\hbar}
\end{aligned}
$$

Where we followed the computations of exercise 0.6.

j

The question is show and explain that $\langle \delta\hat{n}(\mathbf{x}, \tau)\delta\hat{n}(\mathbf{x}', \tau')\rangle$ only depends on r as opposed to on $\mathbf{x}$ and $\mathbf{x}'$.
From h) we know that $\langle \delta\hat{n}(\mathbf{x}, \tau)\delta\hat{n}(\mathbf{x}', \tau')\rangle$ can be written as a product of Greens functions.
$\langle \delta\hat{n}(\mathbf{x}, \tau)\delta\hat{n}(\mathbf{x}', \tau')\rangle = \frac{1}{(\hbar\beta V)^2} \sum_{\mathbf{k},n} \frac{\hbar}{-i\hbar\omega_n + \epsilon_{\mathbf{k}} - \mu} e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')} e^{-i\omega_n(\tau-\tau')} \sum_{\mathbf{k}',n'} \frac{\hbar}{-i\hbar\omega_{n'} + \epsilon_{\mathbf{k}'} - \mu} e^{i\mathbf{k}'(\mathbf{x}'-\mathbf{x})} e^{-i\omega_{n'}(\tau'-\tau)}$
Repeating the steps from g) with the exception of summing over the Matsubara frequencies we can rewrite the $\mathbf{k}$ sum in to an integral and go to polar coordinates. The angle dependence drops likewise.
We should expect this results, because the system is homogenous. The system is thus translation invariant and rotation invariant, which implies that its minimal description depends only on the difference between the space coordinates.

k

$\langle \hat{n}(\mathbf{x}, \mathbf{x})\rangle^2 = n_0^2$. From h) we get that $\langle \delta\hat{n}(\mathbf{x}, \tau)^2\rangle = G(\mathbf{x}, \tau; \mathbf{x}, \tau^+)G(\mathbf{x}, \tau^+; \mathbf{x}, \tau)$. Due to time independence of the system, we can take $\tau' = \tau^+$

$$
\begin{aligned}
\frac{\langle \delta\hat{n}(\mathbf{x}, \tau)^2\rangle}{\langle \hat{n}(\mathbf{x}, \mathbf{x})\rangle^2} &= \lim_{\eta \downarrow 0} \frac{1}{(\hbar\beta V n_0)^2} \sum_{\mathbf{k},n} \frac{e^{-i\omega_n \eta}}{-i\omega_n + (\epsilon_{\mathbf{k}} - \mu)/\hbar} \sum_{\mathbf{k}',n'} \frac{e^{i\omega_n \eta}}{-i\omega_{n'} + (\epsilon_{\mathbf{k}'} - \mu)/\hbar} \\
&= \frac{1}{(V n_0)^2} \sum_{\mathbf{k}} N_B(\epsilon_{\mathbf{k}}) \sum_{\mathbf{k}'} N_B(\epsilon_{\mathbf{k}'}) + 1 \\
&= \frac{1}{V n_0} \sum_{\mathbf{k}} N_B(\epsilon_{\mathbf{k}}) + 1
\end{aligned}
$$

Where we used the results of exercise 7.2 a