Trial Midterm Exam - "Statistical Field Theory"

October 29th, 2013
Duration of the exam: 2 hours

1. Use a separate sheet for every exercise.
2. Write your name and initials in all sheets, on the first sheet also your student ID number.
3. Write clearly, unreadable work cannot be corrected.
4. You are NOT allowed to use any kind of books or lecture notes.

Exercise I - Quantum Ferromagnet: Magnons

The quantum Heisenberg ferromagnet is specified by the Hamiltonian

$$\hat{H} = -J \sum_{\langle mn \rangle} \hat{\mathbf{S}}_m \cdot \hat{\mathbf{S}}_n \tag{1}$$

where $J > 0$, $\hat{\mathbf{S}}_m$ represents the quantum mechanical spin operator at lattice site m, $\langle mn \rangle$ denotes the summation over neighboring sites and $\mathbf{S}_m^2 = S(S+1)$. Holstein and Primakoff have introduced a transformation in which the spin operators $\hat{S}^{\pm}$, $\hat{S}^z$ are specified in terms of bosonic creation and annihilation operators $a^\dagger$ and a:

$$\hat{S}_m^- = a_m^\dagger (2S - a_m^\dagger a_m)^{1/2}, \quad \hat{S}_m^+ = (2S - a_m^\dagger a_m)^{1/2} a_m, \quad \hat{S}_m^z = S - a_m^\dagger a_m. \tag{2}$$

Let us consider the problem in one dimension and put the lattice constant to unity. At low temperatures, for $S \ll 1/2$ we expect the deviations of the magnetization from its value to be very small, i.e. $S - \langle S_m^z \rangle = \langle a_m^\dagger a_m \rangle \ll S$. In this case we may expand $(2S - a_m^\dagger a_m)^{1/2}$ in powers of $a_m^\dagger a_m$.

- (1.0 pt) 1. Show that to first order in $a_m^\dagger a_m / S$ the Heisenberg Hamiltonian takes the form

$$\hat{H} = -JNS^2 + JS \sum_m \left(a_{m+1}^\dagger - a_m^\dagger \right) (a_{m+1} - a_m) + \text{higher order terms} \tag{3}$$

  where N is the total number of lattice sites.

- (1.0 pt) 2. Keeping fluctuations at leading order in S, the quadratic Hamiltonian can be diagonalized by a Fourier transformation. In this case, it is convenient to impose periodic boundary conditions: $\hat{S}_{m+N} = \hat{S}_m$ and $a_{m+N} = a_m$. Perform the Fourier transformation and show that the Hamiltonian takes the form

$$\hat{H} = -JNS^2 + \sum_k \hbar\omega_k a_k^\dagger a_k + \text{higher order terms} \tag{4}$$

where $\hbar\omega_k = 4JS\,\sin^2(k/2)$ represents the dispersion relation of spin excitations. Calculate also the limit $k \to 0$ of the dispersion relation. These massless low-energy excitations, known as magnons, describe the elementary spin-wave excitations of the ferromagnet. Taking into account higher order terms, one finds the interactions between magnons.

Exercise II - Quantum Antiferromagnet: Magnons

The quantum Heisenberg antiferromagnet is specified by the Hamiltonian

$$H = J \sum_{\langle m,n \rangle} \hat{\mathbf{S}}_m \cdot \hat{\mathbf{S}}_n \tag{5}$$

where $J > 0$.
We propose again to study the low-lying excitations of this system using a semi-classical approximation, which amounts to considering the large spin limit $S \gg 1/2$. We focus here on the case of a bipartite lattice, i.e. one that can be separated into two inter-penetrating sub-lattices A and B. In this case, the classical ground state adopts a staggered spin configuration known as the Néel state.

- (1.5 pt) 1. Before studying the fluctuations around this ground state, we apply a canonical transformation in order to rotate the spins of sublattice B by π around the x-axis. Write the transformed spin operators of the B sublattice $\tilde{S}_B$ in terms of the original ones S_B. Deduce the expression of the Hamiltonian of the system in terms of S_A and S_B.

- (1.0 pt) 2. Using the Holstein-Primakoff transformation in the limit of large spins, show that the Hamiltonian takes the form

$$H = -JNS^2 + JS \sum_m \left(a_m^\dagger a_m + a_{m+1}^\dagger a_{m+1} + a_m^\dagger a_{m+1}^\dagger + a_m a_{m+1} \right) + \text{higher order terms} \tag{6}$$

- (1.5 pt) 3. Introduce the Fourier transform for the creation and annihilation operators, and write the Hamiltonian in terms of the latter.

- 4. Show that the Hamiltonian can be diagonalized into

$$H = -JNS^2 + 2JS \sum_k |\sin k| \left(\alpha_k^\dagger \alpha_k + \frac{1}{2} \right). \tag{7}$$

For this, we introduce new operators

$$\alpha_k \equiv A_k a_{-k} + B_k a_k^\dagger, \tag{8}$$

with A_k and B_k as real and even functions of the momentum.

(1.0 pt)(i) Show that in order for the new operators to be bosonic, the functions A_k and B_k have to obey $A_k^2 - B_k^2 = 1$. Show that the vanishing of the off-diagonal terms of the Hamiltonian implies

$$\left[(A_k)^2\cos(k) + (B_k)^2\cos(k) - 2A_kB_k\right] = 0. \tag{9}$$

(1.0 pt)(ii) Solve then for the functions A_k and B_k, and show that the obtained form of the functions leads to Eq. (7).

- (1.0 pt) 5. The difference between a ferromagnet and an antiferromagnet appears when considering the behavior of the magnetization and staggered magnetization (magnetization at one of the subattices) at a finite temperature. The reduction of these observables is given by

$$\Delta M = \alpha \int_0^\Lambda dk n_B(k), \tag{10}$$

where α is a constant,

$$n_B = \frac{1}{\exp\left(\frac{\epsilon(k)}{k_BT}\right) - 1} \tag{11}$$

is the Bose-Einstein distribution, $\epsilon(k)$ is the low-energy dispersion of the magnons, and Λ is the momentum up to which the low-energy approximation is valid. Contrast the behavior of the magnetization and the staggered magnetization at a finite temperature T.

Exercise III (1.0 pt) When you write a coherent state representation for the (ferromagnetic or antiferromagnetic) magnons, do you need to introduce the Grassmann variables? When do you need to use the latter? Explain your answer.