Statistical Field Theory Trial Exam

January 21st, 2014

Duration of the exam: 2 hours

1. *Use a separate sheet for every exercise.*
2. *Write your name and initials in all sheets, on the first sheet also your address and your student ID number.*
3. *Write clearly, unreadable work cannot be corrected.*
4. *You are NOT allowed to use any kind of books or lecture notes.*

Metallic ferromagnetism: Stoner transition

The phase diagram of the lattice Hubbard Hamiltonian is rich, exhibiting a range of correlated ground states, depending on the density and strength of interaction. In the lattice system, commensurability effects can initiate charge- or spin-density wave instabilities, while at large U the electron system can freeze into an insulating antiferromagnetic Mott-Hubbard state. Conversely, at low densities, the system may assume an itinerant (i.e. mobile) spin-polarized phase, the Stoner ferromagnet.

The capacity of the interacting electron-system to form a ferromagnetic phase reflects the competition between the kinetic and interaction potential energies. Being forbidden by the Pauli exclusion principle to occupy the same site, electrons of the same spin can escape the local Hubbard interaction. However, the same exclusion principle requires the system to occupy higher-lying single-particle states, raising the states' kinetic energy. When the total reduction in potential energy outweighs the increase in kinetic energy, a transition to a ferromagnetic phase is induced. This quantum phase transition is the problem we want to study here.

Our starting point is the usual Hubbard Hamiltonian with on-site interaction

$$H = \sum_{\sigma \mathbf{k}} \xi_{\mathbf{k}} c^{\dagger}_{\sigma \mathbf{k}} c_{\sigma \mathbf{k}} + U \sum_{i} \hat{n}_{i\uparrow} \hat{n}_{i\downarrow}$$

where $\xi_{\mathbf{k}} = \epsilon_{\mathbf{k}} - \mu$. The sum runs over N lattice sites i and, as usual, $\hat{n}_{i\sigma} = c^{\dagger}_{i\sigma} c_{i\sigma}$ denotes the number operator for spin σ on site i. We have set $\hbar = 1$. The last term can be recasted in the more convenient expression

$$H_U = \frac{U}{4} \sum_{i} \left(\hat{n}_{i\uparrow} + \hat{n}_{i\downarrow}\right)^2 - \frac{U}{4} \sum_{i} \left(\hat{n}_{i\uparrow} - \hat{n}_{i\downarrow}\right)^2 .$$

The first term is a shift on the chemical potential due to interaction, that can be adsorbed in the definition of μ. The second term can be rewritten as

$$H_U = -U \sum_{i} \left(\vec{S}^z_i\right)^2 ,$$

where $\vec{S}_i^z = (\hat{n}_{i\uparrow} - \hat{n}_{i\downarrow})/2$. In terms of functional field integral, the partition function of the system reads

$$Z = \int D[\psi^*]D[\psi]\exp\Big\{-\int_0^{\hbar\beta} d\tau\Big[\sum_{\mathbf{k}\sigma}\psi^*_{\mathbf{k}\sigma}(\hbar\partial_\tau + \xi_{\mathbf{k}})\psi_{\mathbf{k}\sigma} - \frac{U}{4}\sum_{i\sigma\sigma'}(\psi^*_{i\sigma}\sigma^z_{\sigma\sigma'}\psi_{i\sigma'})^2\Big]\Big\},$$

where ψ is a fermionic field, σ^z is the Pauli-spin matrix, and sum over repeated indices is assumed.

(1.5) **1.** Perform a Hubbard-Stratonovich transformation to decouple the local quartic interaction and show that the partition function becomes

$$\begin{aligned} Z &= \int D[\psi^*]D[\psi]D[m]\exp\Big\{-\int_0^{\hbar\beta} d\tau\Big[\sum_{\mathbf{k}\sigma}\psi^*_{\mathbf{k}\sigma}(\hbar\partial_\tau + \xi_{\mathbf{k}})\psi_{\mathbf{k}\sigma} \\ &- \frac{U}{2}\sum_{i\sigma}(\psi^*_{i\sigma}\sigma^z m_i\psi_{i\sigma}) + \frac{U}{4}\sum_i m_i^2(\tau)\Big]\Big\}. \end{aligned}$$

(1.0) **2.** Noting that

$$\psi_{\mathbf{k}\sigma} = \frac{1}{\sqrt{N}}\sum_j e^{-i\mathbf{k}\cdot\mathbf{r}_j}\psi_{j\sigma},$$

integrate now over the fermionic degrees of freedom to obtain

$$Z = Z_0\int D[m]\exp\Big\{-\frac{U}{4}\int_0^{\hbar\beta} d\tau\sum_i m_i^2(\tau) + \mathrm{Trln}\Big(1 + \frac{U}{2}\sigma^z\hat{m}G_0,\Big)\Big\},$$

where Z_0 and G_0 denote, respectively, the quantum partition function and the Green function of the free electron gas, the matrix $\hat{m} = \{m_i(\tau)\delta_{ij}\delta(\tau - \tau')\}$.

(1.5) **3.** We now want to obtain the effective action for this system to second order in m. First, expanding the $\mathrm{Tr}\ln(1-x) = -\sum_{m=1}^{\infty}\frac{1}{m}\mathrm{Tr}[x^m]$, show that its Fourier-transformed second order term reads

$$-\frac{U^2}{8}\mathrm{Tr}[\sigma^z\hat{m}G_0\sigma^z\hat{m}G_0] = -\frac{U^2}{8N\beta}\sum_{\mathbf{k}\mathbf{q}ns} m_{\mathbf{q},\omega_n}m_{-\mathbf{q},-\omega_n}G_0(\mathbf{k}+\mathbf{q},\omega_{s-n})G_0(\mathbf{k},\omega_s).$$

(1.5) **4.** Now, using the fact that

$$\lim_{\eta\to 0}\frac{1}{\beta}\sum_n\frac{-e^{i\omega_n\eta}}{i\omega_n + \epsilon_{\mathbf{k}} - \mu} = -\frac{1}{e^{\beta(\epsilon_{\mathbf{k}}-\mu)} - 1} - 1 := -n_{\mathrm{F}}(\epsilon_{\mathbf{k}}) - 1,$$

perform the Matsubara summation on the right-hand side of the expression obtained in the previous question to get

$$\mathrm{Tr}[\sigma^z\hat{m}G_0\sigma^z\hat{m}G_0] = -\sum_{\mathbf{q},\omega_n}\Pi_{\mathbf{q},\omega_n}m_{\mathbf{q},\omega_n}m_{-\mathbf{q},-\omega_n},$$

where we have introduced the polarization operator

$$\Pi_{\mathbf{q},\omega_n} = -\frac{1}{N}\sum_{\mathbf{k}} \frac{n_F(\epsilon_{\mathbf{k}}) - n_F(\epsilon_{\mathbf{k}+\mathbf{q}})}{i\omega_n + \xi_{\mathbf{k}} - \xi_{\mathbf{k}+\mathbf{q}}}.$$

(1.5) **5.** Lastly, using these results, show that the action of Z is given to second order by

$$S[m] \approx \frac{1}{2}\sum_{\mathbf{q},\omega_n} \frac{U}{2}\left(1 - \frac{U}{2}\Pi_{\mathbf{q},\omega_n}\right) m_{\mathbf{q},\omega_n} m_{-\mathbf{q},-\omega_n} + \mathcal{O}(m^3)$$

The static susceptibility ($\omega_n = 0$) is dominated by contributions to the momentum where $\xi_{\mathbf{k}} \sim \xi_{\mathbf{k}+\mathbf{q}}$. This involves regions of reciprocal space where $\mathbf{q}$ is small or where, for some non-zero $\mathbf{q} = \mathbf{Q}$, $\xi_{\mathbf{k}} \sim \xi_{\mathbf{k}+\mathbf{q}}$ over a wide region of the Brillouin zone.The second condition reflects nesting symmetry, where a translation by a constant wavevector leaves the spectrum invariant. In this situation, spin-density waves develop. If, instead, the spectrum varies smoothly, so that nesting symmetry is absent, the susceptibility if maximized for $\mathbf{q} = 0$. Assuming now to be close to a ferromagnetic instability, where the ordering vector is $\mathbf{q} = 0$, we want to determine the critical value of the interaction, $U = U_c$, corresponding to the appearance of a non-zero expectation value of the magnetization m_i. By expanding the polarization operator for small frequencies, one eventually finds that the action reads (you don't need to derive it!)

$$S[m] = \sum_{\mathbf{q},\omega_n}\left[r + q^2 + \frac{|\omega_n|}{\Gamma_{\mathbf{q}}}\right] m_{\mathbf{q},\omega_n} m_{-\mathbf{q},-\omega_n} + \frac{u}{4}\int_0^\beta d\tau \int dx (m(x,\tau))^4,$$

where u is a positive constant, $r = (U\nu)^{-1} - 1$, ν is the density of states per volume of the non-interacting electrons at the Fermi surface, and $\Gamma_{\mathbf{q}} = v_F c|\vec{q}|$, with v_F the Fermi velocity, and c a constant that depends on dimensionality (in 3D, $c = \pi/2$).

(1.5) **6.** Knowing that this is a second order phase transition, determine the value of the critical interaction U_c at which a Stoner transition to a spin-polarized phase takes place.

(1.5) **7.** In leading approximation, an understanding of the model can be developed by ignoring the effect of fluctuations - spatial and temporal. In such a mean field approximation, the fields m are assumed to be independent of q or x. The rescaled action then takes the form

$$\frac{S}{\beta L^d} = \frac{r}{2}m^2 + \frac{u}{4}m^4$$

where now $r = U(1 - U\nu)/2$. Determine the behavior of the order parameter at this transition and sketch its shape in a plot.