Final Exam - "Statistical Field Theory"

January 28, 2014
Duration of the exam: 3 hours

1. Use a separate sheet for every exercise.
2. Write your name and initials in all sheets, on the first sheet also your address and your student ID number.
3. Write clearly, unreadable work cannot be corrected.
4. You are NOT allowed to use any kind of books or lecture notes.

Exercise I - Kitaev chain

In 2001 Alexei Kitaev proposed a toy model, referred in the literature as "Kitaev chain", and showed how a $1D$ quantum nanowire can host at its ends a pair of special states precisely at zero energy. Each of these states represents the so-called Majorana fermion, which is a fermion that is its own antiparticle. Kitaev chain is described through a tight-binding Hamiltonian for spinless fermions with p-wave pairing on a one-dimensional lattice, which, in the second quantization, reads

$$H = -\mu \sum_{j=1}^{N} n_j - \sum_{j=1}^{N-1} \left[t \left(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j \right) - \frac{\Delta}{2} \left(c_j c_{j+1} + c_{j+1}^\dagger c_j^\dagger \right) \right] \tag{1}$$

where $c_j^\dagger$ (c_j) is a fermionic operator creating (annihilating) a fermion at lattice site j, and N is the number of lattice sites, assumed to be even. Furthermore, $n_j = c_j^\dagger c_j$ is the corresponding occupation number operator, μ is the chemical potential and $t \geqslant 0$ denotes the hopping amplitude. Finally, $\Delta \geqslant 0$ is the so-called p-wave pairing amplitude. This name stems from the fact that this coupling describes the pairing between fermions with the same spin, and therefore the orbital momentum of a Cooper pair has to be finite, which is equal to one in the case considered here (the same as for the atomic p-orbitals).

(1.0) **1.** Introduce

$$c_j^\dagger = \frac{1}{\sqrt{N}} \sum_k e^{-ikaj} c_k^\dagger, \quad c_j = \frac{1}{\sqrt{N}} \sum_k e^{ikaj} c_k, \tag{2}$$

with a as the distance between two neighboring sites, and $-\frac{\pi m}{aN} \leq k \leq \frac{\pi m}{aN}$ and $|m| \leq N/2$. By using these definitions, show that in the limit of a lattice of infinite length, i.e., $N \to +\infty$, the Hamiltonian takes the form

$$H = \sum_k \epsilon_k c_k^\dagger c_k + \frac{\Delta}{2} \sum_k \left(e^{ika} c_{-k} c_k + e^{-ika} c_k^\dagger c_{-k}^\dagger \right), \tag{3}$$

and give an explicit expression for ϵ_k.

(1.0) **2.** Show that the Hamiltonian (up to an irrelevant constant) can be written as

$$H = \frac{1}{2}\sum_k \begin{pmatrix} c_k^\dagger & c_{-k} \end{pmatrix} \mathbb{H} \begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix}, \tag{4}$$

with $\mathbb{H} = \boldsymbol{\sigma}\cdot\mathbf{d}(\mathbf{k})$, where $d_1(\mathbf{k}) = \Delta\cos ka$, $d_2(\mathbf{k}) = \Delta\sin ka$, $d_3(\mathbf{k}) = -\mu - 2t\cos ka$, and $\boldsymbol{\sigma} \equiv (\sigma_1, \sigma_2, \sigma_3)$ is the vector of Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5}$$

(1.0) **3.** In frequency-momentum representation the action corresponding to the Hamiltonian in Eq. (3) can be written as

$$\begin{aligned} S &= \frac{1}{2}\sum_{k,n} \begin{pmatrix} c_k^\dagger & c_{-k} \end{pmatrix} \begin{pmatrix} -i\hbar\omega_n & 0 \\ 0 & -i\hbar\omega_n \end{pmatrix} \begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix} + H \\ &:= -\frac{1}{2}\sum_{k,n} \begin{pmatrix} c_k^\dagger & c_{-k} \end{pmatrix} \hbar\mathbb{G}^{-1}(k, i\omega_n) \begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix}, \end{aligned} \tag{6}$$

where ω_n are fermionic Matsubara frequencies. Determine the poles of $\mathbb{G}(k,\omega)$ and give their physical interpretation. In particular, comment on the case $\mu = \pm 2t$ and $\mu \neq \pm 2t$. Under which conditions are the quasi-particle excitations gapless?

(1.0) **4.** Now rewrite $\mathbb{G}_{12}(k, i\omega_n)$ in the form

$$\mathbb{G}_{12}(x,\tau;x,\tau^+) = -\frac{\hbar}{\hbar\beta V}\sum_{k,n}\frac{\Delta e^{-ika}}{(\hbar\omega_n)^2 + (\hbar\omega_k)^2}, \tag{7}$$

with $(\hbar\omega_k)^2 \equiv \epsilon_k^2 + \Delta^2$. Perform the summation over Matsubara frequencies to derive

integrate: poles

$$\mathbb{G}_{12}(x,\tau;x,\tau^+) = -\frac{1}{V}\sum_k \frac{\Delta e^{-ika}}{2\hbar\omega_k}\left\{1 - 2N_{\mathrm{FD}}(\hbar\omega_k)\right\}. \tag{8}$$

In the last part of this exercise, we will make a connection between the Kitaev chain and Majorana fermions.

(1.0) **5.** The fermionic creation (annihilation) operator $c_j^\dagger$ (c_j) can be expressed in terms of Majorana-fermion operators γ_j and η_j as

$$c_j = \frac{1}{2}(\eta_j + i\gamma_j), \qquad c_j^\dagger = \frac{1}{2}(\eta_j - i\gamma_j) \tag{9}$$

with $\gamma_j = \gamma_j^\dagger$ and $\eta_j = \eta_j^\dagger$. Determine the anticommutator $\{\eta_j, \gamma_j\}$ and compute η_j^2 and γ_j^2.

(1.0) **6.** Now consider the case in which $\mu = 0$ and $t = \Delta/2 \neq 0$ in Hamiltonian (1). By using the definitions in the previous exercise, introduce Majorana operators for each site j and write down the Hamiltonian (1) in terms of the Majorana operators.

(1.0) **7.** The Hamiltonian found in the previous exercise does not depend on the Majorana fermion operator η_1 at the site $j = 1$ and γ_N at the site $j = N$. What does this suggest?

Answer in text

Exercise II - Superconductivity in graphene

The effective Hamiltonian of graphene is described in terms of two species of *fermions* living on two different sublattices, A and B, of the honeycomb lattice. We will consider the usual term for the nearest-neighbors hopping

$$H_t = -t \sum_{\sigma} \sum_{\langle ij \rangle} a^{\dagger}_{i,\sigma} b_{j,\sigma} + h.c., \tag{10}$$

and the chemical potential term

$$H_{\mu} = -\mu \sum_{i,\sigma} (a^{\dagger}_{i,\sigma} a_{i,\sigma} + b^{\dagger}_{i,\sigma} b_{i,\sigma}), \tag{11}$$

where σ denotes spin up and down. Now, we introduce a local density-density interaction term,

$$H_I = g \sum_i \left(a^{\dagger}_{i,\uparrow} a_{i,\uparrow} a^{\dagger}_{i,\downarrow} a_{i,\downarrow} + b^{\dagger}_{i,\uparrow} b_{i,\uparrow} b^{\dagger}_{i,\downarrow} b_{i,\downarrow} \right). \tag{12}$$

The diagonal form of the non-interacting Hamiltonian ($g = 0$) in the momentum space reads (up to an irrelevant constant)

$$H_0 \equiv H_t + H_{\mu} = \sum_{\mathbf{k}} \Psi^{\dagger}_{\mathbf{k}} \hat{\omega}_{\mathbf{k}} \Psi_{\mathbf{k}}, \tag{13}$$

where $\hat{\omega}_{\mathbf{k}} = \mathrm{diag}(\mu + t|\gamma_{\mathbf{k}}|, \mu - t|\gamma_{\mathbf{k}}|, -\mu + t|\gamma_{\mathbf{k}}|, -\mu - t|\gamma_{\mathbf{k}}|)$ is a diagonal 4×4 matrix, $\gamma_{\mathbf{k}} \equiv \sum_{\boldsymbol{\delta}_j} e^{i\mathbf{k}\cdot\boldsymbol{\delta}_j}$ with $\boldsymbol{\delta}_j$ as the vectors connecting nearest neighboring sites on the honeycomb lattice, and the spinor representation is defined by

$$\Psi_{\mathbf{k}} \equiv \begin{pmatrix} a_{\mathbf{k},\uparrow} \\ b_{\mathbf{k},\uparrow} \\ a^{\dagger}_{-\mathbf{k},\downarrow} \\ b^{\dagger}_{-\mathbf{k},\downarrow} \end{pmatrix}. \tag{14}$$

In order to study superconductivity in graphene, we introduce an order parameter,

$$\Delta = \langle a_{i,\downarrow} a_{i,\uparrow} \rangle = \langle b_{i,\downarrow} b_{i,\uparrow} \rangle. \tag{15}$$

(1.0) **8.** Now consider the total Hamiltonian

$b_{i\sigma} \to \psi_{b,i\sigma}$

$$H = H_t + H_{\mu} + H_I. \tag{16}$$

(a) Replace the operators by fields, e.g., $a_{i,\sigma} \to \psi_{a,i,\sigma}$ and write the corresponding action.

(b) Now, we simplify the notation by dropping the lattice sites "i" and the imaginary time τ from the fields (they are implicit). Using a representation of the unity in the form

$$1 = \int [\mathcal{D}\Delta_a][\mathcal{D}\Delta^*_a][\mathcal{D}\Delta_b][\mathcal{D}\Delta^*_b] \; e^{\frac{1}{\hbar}\int d\tau \sum_i [g(\Delta^*_a + \psi^*_{a,\downarrow}\psi^*_{a,\uparrow})(\Delta_a + \psi_{a,\uparrow}\psi_{a,\downarrow}) + g(\Delta^*_b + \psi^*_{b,\downarrow}\psi^*_{b,\uparrow})(\Delta_b + \psi_{b,\uparrow}\psi_{b,\downarrow})]} \tag{17}$$

eliminate the quartic interaction term, and write down the effective action in terms of the fermion fields $\psi_{a/b,\uparrow/\downarrow}$ and the Hubbard-Stratonovich fields $\Delta_{a/b}$.

$\Delta = \text{const}$ quadratic in fermionic field

eq. 13 $\Rightarrow$ Gaussian integrate

(1.0) **9.** Taking the Hubbard-Stratonovich fields to be constant (spatially and imaginary-time independent), show that after integrating over the fermion fields, the partition function can be written as

Difficult.

$$Z = e^{\mathrm{Tr}\log(-G^{-1})}, \tag{18}$$

with $G^{-1} = G_0^{-1} + \hat{\omega}_{\mathbf{k}} + g\hat{M}(\Delta, \Delta^*)$, where $\omega_{\mathbf{k}}$ was given above,

ω_k given after (13)

$$G_0^{-1} = \mathrm{diag}(-i\hbar\omega_n, -i\hbar\omega_n, i\hbar\omega_n, i\hbar\omega_n), \tag{19}$$

and the matrix $\hat{M}$ has the following form in terms of the Hubbard-Stratonovich fields

$$\hat{M}(\Delta, \Delta^*) = \begin{pmatrix} 0 & 0 & \Delta_a & 0 \\ 0 & 0 & 0 & \Delta_b \\ \Delta_a^* & 0 & 0 & 0 \\ 0 & \Delta_b^* & 0 & 0 \end{pmatrix}. \tag{20}$$

(1.0) **10.** Now we want to study superconductivity in graphene. The action can then be related to the Landau free energy $f_L(|\Delta|)$.

(a) What is the form of f_L? If there is a second order phase transition when the material becomes superconducting, how do the coefficients of the Landau free energy behave at T_c?

(b) Can you relate your answer of the part (a) to the results in exercise **9**? For this you may need the expansion of the logarithm,

$$\log(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}. \tag{21}$$

Do not calculate, just explain it in words.

(c) Sketch $f_L(|\Delta|)$ versus $|\Delta|$ for $T > T_c$ and for $T < T_c$.

no calc

only explain

dcl at end