Ruward

Final Exam - "Statistical Field Theory"

January 27, 2015

1. Use a separate sheet for every exercise.

2. Write your name and initials in all sheets, on the first sheet also your address and your student ID number.

3. Write clearly, unreadable work cannot be corrected.

4. You are NOT allowed to use any kind of books or lecture notes.

Exercise I: Rashba model

We consider a two-dimensional electron gas (2DEG) where (x, y) are coordinates in the two-dimensional plane and z is perpendicular to this plane. If the system does not have inversion symmetry in this latter direction, the appriopiate hamiltonian (ignoring interactions throughout this exercise) is often the so-called Rashba hamiltonian given in first quantization by

$$\hat{H} = \frac{(\hat{p}_x^2 + \hat{p}_y^2)}{2m} + \lambda\,(\boldsymbol{\tau} \times \hat{\mathbf{p}}) \cdot \hat{z}\,, \tag{1}$$

where $\boldsymbol{\tau}$ is a vector of Pauli matrices

$$\tau_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tau_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tau_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2}$$

The term proportional to $\lambda > 0$ is an example of a spin-orbit coupling, i.e., it couples the spin and the momentum of the electron. Ultimately, this term has its origin in relativistic effects.

(0.5) a) Give the hamiltonian in second quantization, i.e., in terms of operators $\hat{\psi}^\dagger_\sigma(\mathbf{x})$ and $\hat{\psi}_\sigma(\mathbf{x})$, where $\mathbf{x} = (x, y)$ and σ denotes the spin (up and down). ?

(0.5) b) We consider the system in the grand-canonical ensemble and use that the partition function is written as a path integral over all anti-periodic Grassmann field evolutions as

$$Z = \int d[\phi^*]d[\phi]e^{-S[\phi^*,\phi]/\hbar}\,.$$

Show that the appropriate Euclidean action $S[\phi^*, \phi]$ is given by

$$\begin{aligned} S[\phi^*,\phi] &= \sum_\sigma \int d\tau \int d\mathbf{x}\, \phi^*_\sigma(\mathbf{x},\tau) \left\{ \hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2\nabla^2}{2m} - \mu \right\} \phi_\sigma(\mathbf{x},\tau) \\ &\quad - \lambda\hbar \int d\tau \int d\mathbf{x} \left[\phi^*_\downarrow(\mathbf{x},\tau) \left\{ i\frac{\partial}{\partial y} + \frac{\partial}{\partial x} \right\} \phi_\uparrow(\mathbf{x},\tau) + \phi^*_\uparrow(\mathbf{x},\tau) \left\{ i\frac{\partial}{\partial y} - \frac{\partial}{\partial x} \right\} \phi_\downarrow(\mathbf{x},\tau) \right]. \end{aligned}$$

(1.0) c) From this action, determine $G^{-1}_{\sigma,\sigma'}(\mathbf{k}, i\omega_n)$. Note that this is not diagonal in spin space.

(1.5) d) From part (c) determine $G_{\sigma,\sigma'}(\mathbf{k}, i\omega_n)$ and write down the expression for $G_{\sigma\sigma'}(\mathbf{x}, \tau; \mathbf{x}', \tau')$ which should be of the following form

$$G_{\sigma\sigma'}(\mathbf{x},\tau;\mathbf{x}',\tau') = -\frac{1}{\hbar\beta V} \sum_{\mathbf{k},n} \frac{\hbar e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}') - i\omega_n(\tau-\tau')}}{(i\hbar\omega_n - \epsilon_{\mathbf{k}} + \mu)^2 - \hbar^2\lambda^2\mathbf{k}^2} A_{\sigma\sigma'}(\mathbf{k}, i\omega_n).$$

To check your result, you should find $A_{\uparrow\uparrow}(\mathbf{k}, i\omega_n) = -i\hbar\omega_n + \epsilon_{\mathbf{k}} - \mu$.

(1.5) e) Calculate $G_{\uparrow\uparrow}(\mathbf{x},\tau;\mathbf{x},\tau^+)$ by performing the sum over the Matsubara frequencies (you don't have to evaluate the sum over $\mathbf{k}$).

(0.5) f) Consider an electron with momentum $\mathbf{p}$, which is described by the Rashba Hamiltonian eq. (1). Along which direction should one pick the quantization axis such that the Rashba Hamiltonian is diagonal in spin-space?

(1.0) g) Calculate the expectation value of the spin density $\langle\mathbf{S}\rangle$, where $\mathbf{S}$ is defined as

$$\mathbf{S} = \sum_{\sigma\sigma'} \phi^*_\sigma(\mathbf{x},\tau^+)\tau_{\sigma\sigma'}\phi_{\sigma'}(\mathbf{x},\tau).$$

Remember that $G_{\sigma\sigma'}(\mathbf{x},\tau;\mathbf{x}',\tau') = -\langle\phi_\sigma(\mathbf{x},\tau)\phi^*_{\sigma'}(\mathbf{x}',\tau')\rangle$

Exercise II: Hubbard-Stratonovich transformation to density and magnetization density

We consider a homogeneous gas of spin one-half particles with short-range interaction potential $V(\mathbf{x}-\mathbf{x}') = V_0\delta(\mathbf{x}-\mathbf{x}')$, with $V_0 > 0$. The action is given by

$$S[\phi^*,\phi] = \int_0^{\hbar\beta} \mathrm{d}\tau \int \mathrm{d}\mathbf{x} \sum_\sigma \phi^*_\sigma(\mathbf{x},\tau)\left[\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2\nabla^2}{2m} - \mu\right]\phi_\sigma(\mathbf{x},\tau)$$
$$+ V_0\int_0^{\hbar\beta}\mathrm{d}\tau\int \mathrm{d}\mathbf{x}\phi^*_\uparrow(\mathbf{x},\tau)\phi^*_\downarrow(\mathbf{x},\tau)\phi_\downarrow(\mathbf{x},\tau)\phi_\uparrow(\mathbf{x},\tau). \tag{3}$$

(0.5) a) Show that the interaction can be written as a sum of a density-density interaction and spin-spin interaction via

$$\phi^*_\uparrow\phi^*_\downarrow\phi_\downarrow\phi_\uparrow = \frac{1}{4}\left(\sum_\sigma \phi^*_\sigma\phi_\sigma\right)^2 - \frac{1}{4}\left(\sum_{\sigma\sigma'}\phi^*_\sigma\tau^z_{\sigma\sigma'}\phi_{\sigma'}\right)^2,$$

where τ^z is the z-th Pauli matrix.

(1.5) b) Perform a Hubbard-Stratonovich transformation and decouple these two contributions to the interactions. In doing so, introduce the density field ρ that is on average equal to $\langle\sum_\sigma \phi^*_\sigma\phi_\sigma\rangle$, and a magnetization density field m_z that is on average determined by $\langle\sum_{\sigma\sigma'}\phi^*_\sigma\tau^z_{\sigma\sigma'}\phi_{\sigma'}\rangle/2$. Give an exact but formal expression for the effective action $S_{\text{eff}}[\rho, m_z]$.

By expanding the effective action around the classical solution one can show that (**do not show**)

$$\langle\rho\rangle = \int\frac{\mathrm{d}\mathbf{k}}{(2\pi)^3}\left[N_F(\epsilon_\mathbf{k} - V_0\langle m_z\rangle - \mu) + N_F(\epsilon_\mathbf{k} + V_0\langle m_z\rangle - \mu)\right], \tag{4}$$
$$\langle m_z\rangle = \frac{1}{2}\int\frac{\mathrm{d}\mathbf{k}}{(2\pi)^3}\left[N_F(\epsilon_\mathbf{k} - V_0\langle m_z\rangle - \mu) - N_F(\epsilon_\mathbf{k} + V_0\langle m_z\rangle - \mu)\right]. \tag{5}$$

A second order phase transition occurs when $\langle m_z\rangle$ takes a non-zero value.

(1.0) c) Using eq. (5), derive an expression from which the critical interaction strength for the phase transition to the ferromagnetic state can be determined. Show that in the zero-temperature limit the equation simplifies to

$$1 = \frac{V_0 m k_F}{2\pi^2\hbar^2}.$$

This is known as the zero-temperature Stoner criterion.

(0.5) d) What can you say about the shape of the Ginzburg-Landau free energy functional and order parameter? Describe what happens at the transition.