Trial Exam Solutions

Exercise I - Rashba model

(1) In the standard spin basis $\sigma \in \{|\uparrow\rangle, |\downarrow\rangle\}$, where $|\uparrow\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, we have that

$$\lambda(\boldsymbol{\tau} \times \hat{\mathbf{p}}) \cdot \hat{z} = \lambda\left(\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\hat{p}_y - \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\hat{p}_x\right). \tag{1}$$

Since $\hat{p}_i = -i\hbar\partial_i$, we can rewrite the Hamiltonian $\hat{H}$ in second quantization as

$$\begin{aligned}\hat{H} =& \sum_\sigma \int d\mathbf{x}\, \hat{\psi}^\dagger_\sigma(\mathbf{x}) \left[-\frac{\hbar^2\left(\partial_x^2 + \partial_y^2\right)}{2m}\right] \hat{\psi}_\sigma(\mathbf{x}) \\ &+\lambda \int d\mathbf{x}\, \left[\hat{\psi}^\dagger_\uparrow(\mathbf{x})(-i\hbar\partial_y + \hbar\partial_x)\hat{\psi}_\downarrow(\mathbf{x}) + \hat{\psi}^\dagger_\downarrow(\mathbf{x})(-i\hbar\partial_y - \hbar\partial_x)\hat{\psi}_\uparrow(\mathbf{x})\right].\end{aligned} \tag{2}$$

(2) We know that the Euclidean action is given by

$$S[\phi^*, \phi] = \sum_\sigma \int d\tau \int d\mathbf{x}\, \phi^*_\sigma(\mathbf{x},\tau)\hbar\frac{\partial}{\partial\tau}\phi_\sigma(\mathbf{x},\tau) + H[\phi^*(\mathbf{x},\tau), \phi(\mathbf{x},\tau)]. \tag{3}$$

Hence,

$$\begin{aligned}S[\phi^*, \phi] =& \sum_\sigma \int d\tau \int d\mathbf{x}\phi^*_\sigma(\mathbf{x},\tau)\left[\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2\nabla^2}{2m} - \mu\right]\phi_\sigma(\mathbf{x},\tau) \\ &-\hbar\lambda \int d\tau \int d\mathbf{x} \left[\phi^*_\downarrow(\mathbf{x},\tau)\left(i\frac{\partial}{\partial y} + \frac{\partial}{\partial x}\right)\phi_\uparrow(\mathbf{x},\tau) + \phi^*_\uparrow(\mathbf{x},\tau)\left(i\frac{\partial}{\partial y} - \frac{\partial}{\partial x}\right)\phi_\downarrow(\mathbf{x},\tau)\right].\end{aligned} \tag{4}$$

(3) We can write

$$S[\phi^*, \phi] = \sum_{\sigma,\sigma'} \int d\tau \int d\tau' \int d\mathbf{x} \int d\mathbf{x}'\, \phi^*_\sigma(\mathbf{x},\tau)\left(-\hbar G^{-1}_{\sigma,\sigma'}(\mathbf{x},\tau;\mathbf{x}',\tau')\right)\phi_{\sigma'}(\mathbf{x}',\tau'). \tag{5}$$

Therefore,

$$\begin{aligned}-\hbar G^{-1}_{\sigma,\sigma'}(\mathbf{x},\tau;\mathbf{x}',\tau') =& \left(\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2\nabla^2}{2m} - \mu\right)\delta_{\sigma,\sigma'}\delta(\mathbf{x}-\mathbf{x}')\delta(\tau-\tau') \\ &+ \lambda\left((\boldsymbol{\tau} \times \hat{\mathbf{p}}) \cdot \hat{z}\right)_{\sigma,\sigma'}\delta(\mathbf{x}-\mathbf{x}')\delta(\tau-\tau').\end{aligned} \tag{6}$$

Using the definition of the inverse, we obtain

$$\delta_{\sigma,\sigma''}\delta(\mathbf{x}-\mathbf{x}'')\delta(\tau-\tau'') = \sum_{\sigma'} \int d\mathbf{x}' \int d\tau' G^{-1}_{\sigma,\sigma'}(\mathbf{x},\tau;\mathbf{x}',\tau')G_{\sigma',\sigma''}(\mathbf{x}',\tau';\mathbf{x}'',\tau''). \tag{7}$$

By substituting the Fourier expansion of $G_{\sigma,\sigma''}(\mathbf{x},\tau;\mathbf{x}'',\tau'')$ given by

$$G_{\sigma,\sigma''}(\mathbf{x},\tau;\mathbf{x}'',\tau'') = \frac{1}{\hbar\beta V}\sum_{\mathbf{k},n} G_{\sigma,\sigma''}(\mathbf{k}, i\omega_n)e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}'')-i\omega_n(\tau-\tau'')}, \tag{8}$$

and using Eq. (6), we can rewrite Eq.(7) into

$$\sum_{\sigma'}\left(-i\hbar\omega_n+\frac{\hbar^2\mathbf{k}^2}{2m}-\mu+\hbar\lambda(\boldsymbol{\tau}\times\mathbf{k})\cdot\hat{z}\right)_{\sigma,\sigma'}G_{\sigma',\sigma''}(\mathbf{k},i\omega_n)=-\hbar\delta_{\sigma,\sigma''}. \tag{9}$$

Hence,

$$M_{\sigma,\sigma'}=\left(-i\hbar\omega_n+\frac{\hbar^2\mathbf{k}^2}{2m}-\mu\right)\delta_{\sigma\sigma'}+(\hbar\lambda(\boldsymbol{\tau}\times\mathbf{k})\cdot\hat{z})_{\sigma,\sigma'}. \tag{10}$$

(4) Since the columns of U are the eigenvectors of M, we have that

$$U^\dagger M U=\begin{pmatrix}\lambda_+ & 0\\ 0 & \lambda_-\end{pmatrix}, \tag{11}$$

with $\lambda_\pm$ the eigenvalues of M. Therefore, the equation for $G(\mathbf{k},i\omega_n)$ can be written as

$$U^\dagger G U=-\hbar\begin{pmatrix}\lambda_+^{-1} & 0\\ 0 & \lambda_-^{-1}\end{pmatrix}, \tag{12}$$

or

$$G=-\hbar U\begin{pmatrix}\lambda_+^{-1} & 0\\ 0 & \lambda_-^{-1}\end{pmatrix}U^\dagger. \tag{13}$$

Thus we need to determine the eigenvalues and eigenvectors of U. The explicit expression for M is

$$M=\begin{pmatrix}\frac{\hbar^2\mathbf{k}^2}{2m}-\mu-i\hbar\omega_n & \hbar\lambda(k_y+ik_x)\\ \hbar\lambda(k_y-ik_x) & \frac{\hbar^2\mathbf{k}^2}{2m}-\mu-i\hbar\omega_n\end{pmatrix}. \tag{14}$$

The eigenvalues are $\lambda_\pm=-i\hbar\omega_n+\frac{\hbar^2\mathbf{k}^2}{2m}-\mu\pm\hbar\lambda|\mathbf{k}|$, and the eigenvectors are

$$\chi_\pm=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ \pm\frac{k_y-ik_x}{|\mathbf{k}|}.\end{pmatrix} \tag{15}$$

Hence

$$U=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ \frac{k_y-ik_x}{|\mathbf{k}|} & \frac{ik_x-k_y}{|\mathbf{k}|}\end{pmatrix}. \tag{16}$$

Therefore,

$$G_{\sigma,\sigma'}(\mathbf{x},\tau;\mathbf{x}',\tau')=-\frac{1}{\hbar\beta V}\sum_{\mathbf{k},n}\frac{\hbar e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')-i\omega_n(\tau-\tau')}}{(i\hbar\omega_n-\epsilon_{\mathbf{k}}+\mu)^2-\hbar^2\lambda^2\mathbf{k}^2}A_{\sigma,\sigma'} \tag{17}$$

where

$$A=\begin{pmatrix}-i\hbar\omega_n+\epsilon_{\mathbf{k}}-\mu & -\hbar\lambda(k_y+ik_x)\\ -\hbar\lambda(k_y-ik_x) & -i\hbar\omega_n+\epsilon_{\mathbf{k}}-\mu\end{pmatrix} \tag{18}$$

(5) $G_{\uparrow\uparrow}(\mathbf{x},\tau;\mathbf{x},\tau^+)$ reads

$$\begin{aligned}G_{\uparrow\uparrow}(\mathbf{x},\tau;\mathbf{x},\tau^+)&=\lim_{\eta\to0}\frac{1}{\hbar\beta V}\sum_{n,\mathbf{k}}\frac{e^{i\omega_n\eta}}{(i\hbar\omega_n-\epsilon_{\mathbf{k}}+\mu)^2-\hbar^2\lambda^2\mathbf{k}^2}(i\hbar\omega_n-\epsilon_{\mathbf{k}}+\mu) \qquad (19)\\ &=\lim_{\eta\to0}\frac{1}{2\hbar\beta V}\sum_{n,\mathbf{k}}\left[\frac{e^{i\omega_n\eta}}{i\hbar\omega_n-\epsilon_{\mathbf{k}}+\mu-\hbar\lambda\mathbf{k}}+\frac{e^{i\omega_n\eta}}{i\hbar\omega_n-\epsilon_{\mathbf{k}}+\mu+\hbar\lambda\mathbf{k}}\right].\end{aligned}$$

Now define $\hbar\omega_k = \epsilon_{\mathbf{k}} - \mu + \hbar\lambda\mathbf{k}$ and $\hbar\tilde{\omega}_k = \epsilon_{\mathbf{k}} - \mu - \hbar\lambda\mathbf{k}$, and we rewrite Eq.(19) as

$$G_{\uparrow\uparrow}(\mathbf{x},\tau;\mathbf{x},\tau^+) = \lim_{\eta\to 0}\frac{1}{2\hbar\beta V}\sum_{n,\mathbf{k}}\left[\frac{e^{i\omega_n\eta}}{i\hbar\omega_n - \hbar\omega_k} + \frac{e^{i\omega_n\eta}}{i\hbar\omega_n - \hbar\tilde{\omega}_k}\right]. \tag{20}$$

Now we perform the sum over Matsubara frequencies by using contour integration. Here we focus on the first term. If C is a contour fully enclosing the imaginary axis, then Cauchy's theorem (or maybe better: the residue theorem) tells us that

$$\lim_{\eta\downarrow 0}\frac{1}{2\pi i}\int_C dz\,\frac{e^{\eta z}}{z-\omega_k}\frac{-1}{e^{\hbar\beta z}+1} = \lim_{\eta\downarrow 0}\frac{1}{\hbar\beta}\sum_n\frac{e^{i\omega_n\eta}}{i\omega_n - \omega_k}, \tag{21}$$

Now, we can freely add the curves C' as shown in Fig. 7.2 of Stoof, since the integral over these curves with infinite radius vanishes. As a result we have two closed contours, both half circles. We call the left half circle C_L and the right half circle C_R. This yields

$$\begin{aligned}\lim_{\eta\downarrow 0}\frac{1}{2\pi i}\int_C dz\,\frac{e^{\eta z}}{z-\omega_k}\frac{-1}{e^{\hbar\beta z}+1} &= \lim_{\eta\downarrow 0}\frac{1}{2\pi i}\int_{C_L} dz\,\frac{e^{\eta z}}{z-\omega_k}\frac{-1}{e^{\hbar\beta z}+1}\\ &+ \lim_{\eta\downarrow 0}\frac{1}{2\pi i}\int_{C_R} dz\,\frac{e^{\eta z}}{z-\omega_k}\frac{-1}{e^{\hbar\beta z}+1}.\end{aligned} \tag{22}$$

Note that only C_R encloses a pole at $z=\omega_k$, and the other contour integral does not enclose a pole and therefore gives 0. By using the residue theorem and taking the limit of η to zero, we find

$$\lim_{\eta\downarrow 0}\frac{1}{2\pi i}\int_{C_R}\frac{e^{\eta z}}{-z-\omega_k}\frac{-1}{e^{\hbar\beta z}+1} = \frac{1}{e^{\beta\hbar\omega_k}+1} := N_F(\omega_k), \tag{23}$$

where there is extra minus sign from the fact that the contour C_L is clockwise instead of counterclockwise. Hence,

$$G_{\uparrow\uparrow}(\mathbf{x},\tau;\mathbf{x},\tau^+) = \frac{1}{2V}\sum_{\mathbf{k}}\left(N_F(\omega_k) + N_F(\tilde{\omega}_k)\right) \tag{24}$$

(6) We can see from $\lambda(\boldsymbol{\tau}\times\hat{\mathbf{p}})\cdot\hat{z} \propto \lambda(\hat{z}\times\hat{\mathbf{p}})\cdot\boldsymbol{\tau}$ that the orientation of the spin of an electron with momentum $\mathbf{p}$ lies in the x-y plane and is perpendicular to $\mathbf{p}$. This constrains the spin of an electron having momentum $-\mathbf{p}$ to be oriented exactly in the opposite direction. Hence, by averaging over momentum, the expectation value of the spin density is zero.

(7) Let us calculate the expectation value of the spin density as

$$\begin{aligned}\langle\mathbf{S}\rangle &= \sum_{\sigma,\sigma'}\langle\phi^*_\sigma(\mathbf{x},\tau^+)\boldsymbol{\tau}_{\sigma,\sigma'}\phi_{\sigma'}(\mathbf{x},\tau)\rangle\\ &= \left(\langle\phi^*_\uparrow\phi_\downarrow\rangle + \langle\phi^*_\downarrow\phi_\uparrow\rangle\right)\hat{x} + i\left(\langle\phi^*_\downarrow\phi_\uparrow\rangle - \langle\phi^*_\uparrow\phi_\downarrow\rangle\right)\hat{y} + \left(\langle\phi^*_\uparrow\phi_\uparrow\rangle - \langle\phi^*_\downarrow\phi_\downarrow\rangle\right)\hat{z}\\ &= -\left(G_{\downarrow\uparrow}(\mathbf{x},\tau;\mathbf{x},\tau^+) + G_{\uparrow\downarrow}(\mathbf{x},\tau;\mathbf{x},\tau^+)\right)\hat{x} + i\left(G_{\uparrow\downarrow}(\mathbf{x},\tau;\mathbf{x},\tau^+) - G_{\downarrow\uparrow}(\mathbf{x},\tau;\mathbf{x},\tau^+)\right)\hat{y}\\ &- \left(G_{\uparrow\uparrow}(\mathbf{x},\tau;\mathbf{x},\tau^+) - G_{\downarrow\downarrow}(\mathbf{x},\tau;\mathbf{x},\tau^+)\right)\hat{z}.\end{aligned} \tag{25}$$

If we now look at the result of part 4), we notice that $G_{\uparrow\uparrow}(\mathbf{x},\tau;\mathbf{x},\tau^+) = G_{\downarrow\downarrow}(\mathbf{x},\tau;\mathbf{x},\tau^+)$ and $G_{\uparrow\downarrow}(\mathbf{x},\tau;\mathbf{x},\tau^+) = G_{\downarrow\uparrow}(\mathbf{x},\tau;\mathbf{x},\tau^+) = 0$ (since we sum an odd function in $\mathbf{k}$ over all values of $\mathbf{k}$). Hence, $\langle\mathbf{S}\rangle = 0$.

Exercise II - Hubbard-Stratonovich transformation to density and magnetization density

(8) We have

$$\left(\sum_{\sigma}\phi^*_{\sigma}\phi_{\sigma}\right)^2-\left(\sum_{\sigma,\sigma'}\phi^*_{\sigma}\tau^z_{\sigma\sigma'}\phi_{\sigma'}\right)^2=\left(\phi^*_{\uparrow}\phi_{\uparrow}+\phi^*_{\downarrow}\phi_{\downarrow}\right)^2-\left(\phi^*_{\uparrow}\phi_{\uparrow}-\phi^*_{\downarrow}\phi_{\downarrow}\right)^2 \tag{26}$$

$$=2\phi^*_{\uparrow}\phi_{\uparrow}\phi^*_{\downarrow}\phi_{\downarrow}+2\phi^*_{\downarrow}\phi_{\downarrow}\phi^*_{\uparrow}\phi_{\uparrow}=4\phi^*_{\uparrow}\phi^*_{\downarrow}\phi_{\downarrow}\phi_{\uparrow}.$$

Hence,

$$\phi^*_{\uparrow}\phi^*_{\downarrow}\phi_{\downarrow}\phi_{\uparrow}=\frac{1}{4}\left(\sum_{\sigma}\phi^*_{\sigma}\phi_{\sigma}\right)^2-\frac{1}{4}\left(\sum_{\sigma,\sigma'}\phi^*_{\sigma}\tau^z_{\sigma\sigma'}\phi_{\sigma'}\right)^2. \tag{27}$$

(9) In order to decouple the two contributions to the interactions, we have to perform two Hubbard-Stratonovich transformations and cancell the two interaction terms. Therefore, we introduce a density field ρ that is on average equal to $\langle\sum_{\sigma}\phi^*_{\sigma}\phi_{\sigma}\rangle$, and a magnetization density field m_z that is on average determined by $\langle\sum_{\sigma,\sigma'}\phi^*_{\sigma}\tau^z_{\sigma\sigma'}\phi_{\sigma'}\rangle$. Thus, we write 1 as

$$1=\int d[\rho]\,e^{\int_0^{\hbar\beta}d\tau\int d\mathbf{x}\left(\rho(\mathbf{x},\tau)-\sum_{\sigma}\phi^*_{\sigma}\phi_{\sigma}\right)\frac{V_0}{4\hbar}\left(\rho(\mathbf{x},\tau)-\sum_{\sigma}\phi^*_{\sigma}\phi_{\sigma}\right)}. \tag{28}$$

and

$$1=\int d[m_z]\,e^{\int_0^{\hbar\beta}d\tau\int d\mathbf{x}\left(m_z(\mathbf{x},\tau)-\frac{1}{2}\sum_{\sigma,\sigma'}\phi^*_{\sigma}\tau^z_{\sigma\sigma'}\phi_{\sigma'}\right)\frac{V_0}{\hbar}\left(m_z(\mathbf{x},\tau)-\frac{1}{2}\sum_{\sigma,\sigma'}\phi^*_{\sigma}\tau^z_{\sigma\sigma'}\phi_{\sigma'}\right)}. \tag{29}$$

This leads to

$$\begin{aligned}Z&:=\int d[\phi^*]\int d[\phi]\exp\{-S[\phi^*,\phi]/\hbar\} \qquad (30)\\&=\int d[\phi^*]\int d[\phi]\int d[\rho]\int d[m_z]\exp\left\{\frac{V_0}{4\hbar}\left(\int_0^{\hbar\beta}d\tau\int d\mathbf{x}\,\left[\rho^2(\mathbf{x},\tau)-4m_z^2(\mathbf{x},\tau)\right]\right)\right\}\\&\quad\times\exp\left\{\int_0^{\hbar\beta}d\tau\int d\mathbf{x}\sum_{\sigma}\phi^*_{\sigma}(\mathbf{x},\tau)\left[G^{-1}_{0,\sigma}(\mathbf{x},\tau)-\Sigma_{\rho}(\mathbf{x},\tau)+\Sigma^{\sigma}_{\mathrm{m}}(\mathbf{x},\tau)\right]\phi_{\sigma}(\mathbf{x},\tau)\right\},\end{aligned}$$

where we defined $-\hbar G^{-1}_{0,\sigma}(\mathbf{x},\tau)=\hbar\frac{\partial}{\partial\tau}-\frac{\hbar^2\nabla^2}{2m}-\mu$, $\hbar\Sigma_{\rho}(\mathbf{x},\tau)=\frac{V_0}{2}\rho(\mathbf{x},\tau)$ and $\hbar\Sigma^{\sigma}_{\mathrm{m}}(\mathbf{x},\tau)=\delta_{\sigma}V_0m_z(\mathbf{x},\tau)$. Furthermore, $\delta_{\uparrow}=1$ and $\delta_{\downarrow}=-1$. By performing the integration over ϕ^*_{σ} and ϕ_{σ}, we obtain

$$\begin{aligned}Z&=\exp\left\{\sum_{\sigma}\mathrm{Tr}\left[\mathrm{Log}\left(-G^{-1}_{0,\sigma}(\mathbf{x},\tau)+\Sigma_{\rho}(\mathbf{x},\tau)+\Sigma^{\sigma}_{\mathrm{m}}(\mathbf{x},\tau)\right)\right]\right\} \qquad (31)\\&\quad\times\int d[\rho]\int d[m_z]\exp\left\{\frac{V_0}{4\hbar}\left(\int_0^{\hbar\beta}d\tau\int d\mathbf{x}\,\left[\rho^2(\mathbf{x},\tau)-4m_z^2(\mathbf{x},\tau)\right]\right)\right\}\\&:=\int d[\rho]\int d[m_z]\exp\left\{-S_{\mathrm{eff}}[\rho,m_z]/\hbar\right\},\end{aligned}$$

where

$$S_{\rm eff}[\rho, m_z] = -\hbar \sum_{\sigma} {\rm Tr}\left[{\rm Log}\left(-G_{0,\sigma}^{-1}(\mathbf{x},\tau) + \Sigma_\rho(\mathbf{x},\tau) + \Sigma_{\rm m}^{\sigma}(\mathbf{x},\tau)\right)\right] + \frac{V_0}{4}\left(\int_0^{\hbar\beta} d\tau \int d\mathbf{x}\, \left[4m_z^2(\mathbf{x},\tau) - \rho^2(\mathbf{x},\tau)\right]\right). \tag{32}$$

(10) We are interested in the regime close to the phase transition, and therefore we assume that $\langle m_z\rangle$ is small. By making a Taylor expansion into the equation for $\langle m_z\rangle$, we obtain

$$\langle m_z\rangle = -\int \frac{d\mathbf{k}}{(2\pi)^3} V_0 \langle m_z\rangle \frac{\partial N_{\rm FD}(\epsilon_{\mathbf{k}}-\mu)}{\partial(\epsilon_{\mathbf{k}}-\mu)}. \tag{33}$$

Thus, the equation from which the critical temperature and interaction strength for the phase transition to the ferromagnetic state can be determined, reads

$$-\int \frac{d\mathbf{k}}{(2\pi)^3} \frac{\partial N_{\rm FD}(\epsilon_{\mathbf{k}}-\mu)}{\partial(\epsilon_{\mathbf{k}}-\mu)} = \frac{1}{V_0}. \tag{34}$$

In the zero-temperature limit $N_{\rm FD}(x) = \Theta(-x)$. Hence, in the zero-temperature limit

$$\begin{aligned} \frac{1}{V_0} &= \int \frac{d\mathbf{k}}{(2\pi)^3} \delta(\epsilon_{\mathbf{k}}-\mu) = \frac{1}{2\pi^2}\int_0^\infty dk\, k^2 \delta(\hbar^2k^2/2m - \mu) \\ &= \frac{1}{2\pi^2}\int_0^\infty dk\, k^2 \sqrt{\frac{m}{2\hbar^2\mu}}\left(\delta(k-\sqrt{2m\mu}/\hbar) + \delta(k+\sqrt{2m\mu}/\hbar)\right) \\ &= \frac{m}{2\pi^2\hbar^2}\sqrt{\frac{2m\mu}{\hbar^2}} = \frac{mk_{\rm F}}{2\pi^2\hbar^2}. \end{aligned} \tag{35}$$

Therefore, in the zero-temperature limit the equation for the critical interaction strength for the phase transition is given by

$$1 = \frac{V_0 m k_{\rm F}}{2\pi^2\hbar^2}. \tag{36}$$

(11) The Ginzburg-Landau free energy functional is given by

$$F_{\rm L} = \alpha(T)\langle m_z\rangle^2 + \beta(T)\langle m_z\rangle^4 + \delta(T)\langle m_z\rangle^6 + ... \tag{37}$$

Here we assume that there is a second order phase transition. Then, above the critical temperature all coefficients are positive such that there is a minimum at $\langle m_z\rangle = 0$. However, if we lower the temperature $\alpha(T)$ becomes negative, and the minimum of Ginzburg-Landau free energy functional shifts away in a continuous manner to a nonzero value of $\langle m_z\rangle$.