Midterm Exam - "Statistical Field Theory"

November 7, 2017, 13:30 - 16:30

1. For every exercise, it is indicated how many points it is worth.
2. Write your name and initials on every sheet, on the first sheet also your address and your student ID number.
3. Write clearly, unreadable work cannot be corrected.
4. You are NOT allowed to use any kind of books or lecture notes.

Kitaev chain and spin models

1 Periodic Kitaev chain

Consider the following Hamiltonian for interacting spinless fermions in 1D

$$\hat{H} = -\mu \sum_j c_j^\dagger c_j - t \sum_j \left(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j \right) + U \sum_j c_j^\dagger c_{j+1}^\dagger c_{j+1} c_j, \tag{1}$$

where c_j ($c_j^\dagger$) are the fermionic annihilation (creation) operators, μ the chemical potential, t the hopping parameter and U the interaction strength.

1. (1.0) We will deal with the interaction term by applying a mean-field approximation with the mean-field parameters $\Delta_j = U\langle c_{j+1} c_j \rangle$ and $\Delta_j^* = U\langle c_j^\dagger c_{j+1}^\dagger \rangle$. Assume that the mean-field parameter Δ_j is position independent and real, i.e. $\Delta \equiv \Delta_j = \Delta_j^*$. Show that if you ignore a constant term, the Hamiltonian reads

$$\hat{H} = -\mu \sum_j c_j^\dagger c_j - t \sum_j \left(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j \right) + \Delta \sum_j \left(c_j^\dagger c_{j+1}^\dagger + c_{j+1} c_j \right). \tag{2}$$

   This is the well-known Kitaev chain model.

2. (0.5) Show that, by applying a Fourier transformation, the Hamiltonian reads

$$\hat{H} = -\sum_k \epsilon_k c_k^\dagger c_k + \sum_k \Delta_k (c_k^\dagger c_{-k}^\dagger - c_{-k} c_k), \tag{3}$$

   where $\epsilon_k = \mu + 2t \cos k$ and $\Delta_k = i\Delta \sin k$.

3. (0.25) a) Is this Hamiltonian Hermitian? Justify your answer.
(0.75) b) Which phenomenon are you describing with Eq. (3) and the mean-field approximation that you performed? Look at Δ_k and explain.

4. (0.5) One can diagonalize this Hamiltonian for a periodic chain by applying a Bogoliubov transformation of the form

$$c_k = \alpha_k d_k + \beta_k d^\dagger_{-k}$$

where $\alpha_k, \beta_k \in \mathbb{C}$ and we impose that d_k ($d_k^\dagger$) are fermionic annihilation (creation) operators. Use the anti-commutation relations to find the following constraints on the coefficients α and β

$$|\alpha_k|^2 + |\beta_k|^2 = 1 \tag{4}$$
$$\alpha_k \beta_{-k} + \beta_k \alpha_{-k} = 0.$$

5. (1.0) Let us choose $\alpha_k = \alpha_{-k}$ and $\beta_k = -\beta_{-k}$, such that the Bogoliubov transformation is given by

$$c_k = \alpha_k d_k + \beta_k d^\dagger_{-k}$$
$$c_{-k} = \alpha_k d_{-k} - \beta_k d^\dagger_k.$$

Apply the Bogoliubov transformation on the Hamiltonian in Eq. (3) and find the conditions for the Hamiltonian to be diagonal in the operators d_k.

6. (0.5) To solve for α_k and β_k, get a quadratic equation in α_k^*/β_k and show that

$$\frac{\alpha_k^*}{\beta_k} = \frac{-\epsilon_k \pm \sqrt{\epsilon_k^2 + 4\Delta_k^2}}{2\Delta_k}. \tag{5}$$

7. (0.5) Choose α_k real and show that, by using Eq. (4) and Eq. (5), a possible solution is given by

$$\alpha_k = \frac{\sqrt{E_k + \epsilon_k}}{\sqrt{2E_k}},$$
$$\beta_k = \frac{-2\Delta_k}{\sqrt{2E_k}\sqrt{E_k + \epsilon_k}},$$

where $E_k^2 = \epsilon_k^2 + 4\Delta_k^2$.

8. The Hamiltonian can now be written as

$$\hat{H} = \sum_k E_k d_k^\dagger d_k + \text{constant}. \tag{6}$$

(0.5) a) Determine the constant and describe its physical interpretation.
(0.5) b) Describe the physical meaning of E_k.
(0.5) c) Find the values of μ, Δ and t for which $E_k = 0$ is possible. At those points, one finds a (topological) phase transition.

2 Jordan-Wigner transformation

Now, you are going to apply a Jordan-Wigner transformation to rewrite the Kitaev chain as a spin model. This allows you to interpret the phase transition you found before in a different way.

9. (1.0) Apply the Jordan Wigner transformation on Eq. (2) to obtain the Hamiltonian

$$\hat{H} = -\mu \sum_j \left[S_j^z + \frac{1}{2} \right] - \sum_j \left[2(t-\Delta) S_{j+1}^x S_j^x + 2(t+\Delta) S_{j+1}^y S_j^y \right]. \tag{7}$$

   Recall that the Jordan-Wigner transformation reads

$$c_j = e^{i\phi_j} S_j^-,$$
$$c_j^\dagger = e^{-i\phi_j} S_j^+,$$

   where $S_j^\pm$ is the spin raising/lowering operator on site j and

$$\phi_j = \pi \sum_{l=1}^{j-1} \left(\frac{1}{2} + S_l^z \right) = \pi \sum_{l=1}^{j-1} c_l^\dagger c_l.$$

10. (0.25) What kind of model is the Hamiltonian in Eq. (7) describing? How is this model named?

11. Consider the competition between the transverse field (first term) and the nearest-neighbour interactions (second term) for $\Delta = t$ and describe the ground state and the ground state energy in the limits

    (0.25) a) $|\mu/t| >> 1$, $\mu > 0$,
    (0.25) b) $|\mu/t| >> 1$, $\mu < 0$,

    (0.25) c) $|t/\mu| >> 1$, $t > 0$,
    (0.25) d) $|t/\mu| >> 1$, $t < 0$.

    (0.25) e) For which values of μ and t do you expect a phase transition in this model? How does this relate to your answer of question 8?

3 Phase Transtition in the Finite Kitaev Chain

Until now, you considered a periodic chain. A similar, but more elaborate, diagonalization procedure can be applied on the finite chain to get the diagonal Hamiltonian

$$\hat{H} = \sum_{j=1}^{N} \mathcal{E}_j d_j^\dagger d_j, \tag{8}$$

where N denotes the number of sites and the $\mathcal{E}_j$ are the eigenenergies. We ignored the constant for convenience.
The topological phase transition mentioned in question 8 is most apparent in the finite chain, due to the presence of edge states. This topological phase transition can be detected by a non-analyticity in one of the thermodynamic identities. Here, you will show that there will be a jump in the entropy S at zero temperature when going from one phase to the other, i.e. from $|\mu/t| < 2$ to $|\mu/t| > 2$. To remind you, the free energy F can be calculated from the partition function Z through

$$F = -k_B T \log Z,$$

where k_B is the Boltzmann constant and T the temperature. Additionally, the thermodynamic identity reads $\mathrm{d}F = -S\mathrm{d}T - p\mathrm{d}V$, where p denotes the pressure and V the volume (i.e. length in 1D).

12. (0.5) Calculate the partition function (at finite T) for the diagonalized Hamiltonian of the finite chain given in Eq. (8) and use this to derive an expression for the entropy S at finite temperatures.

13. (0.5) In the topological phase $|\mu/t| < 2$, one of the energies for the finite chain is zero (i.e. there is one j for which $\mathcal{E}_j = 0$), while all the energies in the trivial phase $|\mu/t| > 2$ are strictly positive (i.e. $\mathcal{E}_j > 0$ for all j). Take the limit of $T \to 0$ in both phases and show that $S = \log(2)$ in the topological phase $|\mu/t| < 2$ and $S = 0$ in the trivial phase $|\mu/t| > 2$.

For further clarification: In general, the entropy at zero temperature is given by the logarithm of the degeneracy of the ground state. This means in our case, that the ground state is double degenerate in the topological phase and non-degenerate in the trivial phase. This degeneracy in the topological phase is due to the Majorana modes at the ends of the finite chain. Hence, in the topological phase we find edge states and in the trivial we do not.