Institute for Theoretical Physics
Utrecht University

Midterm Exam "Statistical Field Theory"

Tuesday, November 6, 2018

Duration of the exam: 3 hours

1. *Use a seperate sheet for every excercise.*
2. *Write your name and initials on all sheets, on the first sheet also your address and your student ID number.*
3. *Write clearly, unreadable work cannot be corrected.*
4. *You may use the book "Ultracold Quantum Fields".*

Excercise: Spin-1/2 particles.

Consider a many-body system of spin-1/2 particles. The spin operators of this system are, in second-quantized language, given by

$$\hat{\mathbf{S}} = \frac{\hbar}{2} \sum_{\alpha,\alpha'} \int d\mathbf{x}\ \hat{\psi}^\dagger_\alpha(\mathbf{x}) \boldsymbol{\sigma}_{\alpha,\alpha'} \hat{\psi}_{\alpha'}(\mathbf{x}) \ , \tag{1}$$

where $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the vector of Pauli matrices. So

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ . \tag{2}$$

a) Show that these spin operators obey the desired commutation relations

$$[\hat{S}_x, \hat{S}_y]_- = i\hbar \hat{S}_z \ , \quad [\hat{S}_y, \hat{S}_z]_- = i\hbar \hat{S}_x \ , \quad [\hat{S}_z, \hat{S}_x]_- = i\hbar \hat{S}_y \ , \tag{3}$$

both for bosonic as well as fermionic field operators. (Note that in condensed-matter we can also have spin-1/2 bosons, so Pauli matrices can also be used in the bosonic case!)

Consider next the Zeeman interaction with a magnetic field B, associated with the Hamiltonian

$$\hat{H} = -\gamma B \sum_{\alpha,\alpha'} \int d\mathbf{x}\ \hat{\psi}^\dagger_\alpha(\mathbf{x}) (\sigma_z)_{\alpha,\alpha'} \hat{\psi}_{\alpha'}(\mathbf{x}) \ , \tag{4}$$

b) Derive the Heisenberg equations of motion for the field operators $\hat{\psi}_\alpha(\mathbf{x}, t)$ and $\hat{\psi}^\dagger_\alpha(\mathbf{x}, t)$, respectively. Solve these equations in terms of the initial operators $\hat{\psi}_\alpha(\mathbf{x}, 0)$ and $\hat{\psi}^\dagger_\alpha(\mathbf{x}, 0)$ at $t = 0$.

c) Derive also the Heisenberg equations of motion for $\hat{\mathbf{S}}(t)$ and solve these equations in terms of the initial operators $\hat{\mathbf{S}}(0)$ at $t = 0$.

d) Show that the solutions obtained in items b) and c) are consistent with each other, i.e., show that these solutions obey equation (1) at all times.

Finally, we imagine that our experimental friends are able to prepare a single particle exactly into a certain spatial state with wavefunction $\chi(\mathbf{x})$. Consider next the two-particle initial state $|\Psi_{\alpha,\alpha'}(0)\rangle$, in which both particles have the same spatial wavefunction $\chi(\mathbf{x})$, but possible different spins states $|\alpha\rangle$ and $|\alpha'\rangle$, respectively, that are eigenstates of σ_z.

e) Give in second quantization, and both for bosonic as well as fermionic particles, the time-dependent wavefunctions $|\Psi_{\alpha,\alpha'}(t)\rangle$ and determine from this expression the energies of all the possible two-particle states.