SFT midterm exam

$$\hat{S} = \frac{\hbar}{2}\sum_{\alpha,\alpha'}\int dx\; \hat{\psi}^\dagger_\alpha(x)\;\sigma_{\alpha\alpha'}\;\hat{\psi}_{\alpha'}(x)$$

a) $$[S_i, S_j] = \frac{\hbar^2}{4}\sum_{\alpha\alpha'}\sum_{\beta\beta'}\int dx\int dx' \left[\psi^\dagger_\alpha(x)\,\sigma^i_{\alpha\alpha'}\,\hat{\psi}_{\alpha'}(x),\; \psi^\dagger_\beta(x')\,\sigma^j_{\beta\beta'}\,\psi_{\beta'}(x')\right]$$

$$= \frac{\hbar^2}{4}\sum_{\alpha\alpha'}\sum_{\beta\beta'}\int dx\int dx'\,\sigma^i_{\alpha\alpha'}\,\sigma^j_{\beta\beta'}\left[\psi^\dagger_\alpha(x)\psi_{\alpha'}(x),\; \psi^\dagger_\beta(x')\psi_{\beta'}(x')\right] =$$

$$= \frac{\hbar^2}{4}\sum_{\alpha\alpha'}\sum_{\beta\beta'}\int dx\int dx'\;\sigma^i_{\alpha\alpha'}\,\sigma^j_{\beta\beta'}\Big(\psi^\dagger_\alpha(x)\psi_{\alpha'}(x)\psi^\dagger_\beta(x')\psi_{\beta'}(x') \;+$$

$$- \underbrace{\psi^\dagger_\beta(x')\psi_{\beta'}(x')\,\psi^\dagger_\alpha(x)\psi_{\alpha'}(x)}_{=\;1}\Big)$$

$$[\psi_\alpha(x), \psi^\dagger_\beta(x')]_\mp = \delta_{\alpha\beta}\,\delta(x-x')$$

$$1 = \psi^\dagger_\beta(x')\left(\delta_{\alpha\beta'}\,\delta(x-x') \pm \psi^\dagger_\alpha(x)\psi_{\beta'}(x')\right)\psi_{\alpha'}(x)$$

$$= \psi^\dagger_\beta(x')\,\delta_{\alpha\beta'}\,\delta(x-x')\,\psi_{\alpha'}(x) \pm \psi^\dagger_\alpha(x)\psi^\dagger_\beta(x')\,\psi_{\alpha'}(x)\,\psi_{\beta'}(x') =$$

$$(\pm 1 \times \pm 1)$$

$$= \psi^\dagger_\beta(x')\,\delta_{\alpha\beta'}\,\delta(x-x')\,\psi_{\alpha'}(x) - \underline{\psi^\dagger_\alpha(x)\left(\delta_{\alpha'\beta}\,\delta(x-x') - \psi_{\alpha'}(x)\psi^\dagger_\beta(x')\right)\psi_{\beta'}(x')}$$

here I used:
$$\mp\psi^\dagger_\beta(x')\psi_{\alpha'} = \delta_{\alpha'\beta}\,\delta(x-x') - \psi_{\alpha'}(x)\psi^\dagger_\beta(x')$$

$$\Rightarrow [S_i, S_j] = \frac{\hbar^2}{4}\sum_{\alpha\alpha'}\int dx\Big(-\sum_\beta \psi^\dagger_\beta(x)\,\sigma^j_{\beta\alpha}\,\sigma^i_{\alpha\alpha'}\,\psi_{\alpha'}(x) \;+$$

$$+ \sum_\beta \psi^\dagger_\alpha(x)\,\sigma^i_{\alpha\beta}\,\sigma^j_{\beta\alpha'}\,\psi_{\alpha'}(x)\Big) =$$

$$= \frac{\hbar^2}{4}\sum_{\alpha\alpha'}\int dx\;\psi^\dagger_\alpha(x)\left(\sum_\beta \sigma^i_{\alpha\beta}\,\sigma^j_{\beta\alpha'} - \sum_\beta \sigma^j_{\alpha\beta}\,\sigma^i_{\beta\alpha'}\right)\psi_{\alpha'}(x)$$

$$= \frac{\hbar^2}{4}\sum_{\alpha\alpha'}\int dx\;\psi^\dagger_\alpha(x)\,[\sigma^i, \sigma^j]_{\alpha\alpha'}\,\psi_{\alpha'}(x)$$

$$[\sigma^i, \sigma^j] = 2i\,\sigma^k \varepsilon_{ijk} \qquad \text{with} \quad i,j,k \in \{x,y,z\}$$

$$\Rightarrow \quad [S_i, S_j] = i\hbar\,\varepsilon_{ijk}\left(\frac{\hbar}{2}\sum_{\alpha\alpha'}\int dx\, \psi^\dagger_\alpha(x)\, \sigma^k_{\alpha\alpha'}\psi_{\alpha'}(x)\right) = i\hbar\,\varepsilon_{ijk} S_k$$

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b. $$\hat{H} = -\gamma B \sum_{\alpha\alpha'} \int dx\, \hat{\varphi}^{+}_{\alpha}(x)\, \sigma^{z}_{\alpha\alpha'}\, \hat{\varphi}_{\alpha'}(x)$$

Heisenberg's picture:

$$\hat{O}(t) = e^{i\hat{H}t/\hbar}\, \hat{O}\, e^{-i\hat{H}t/\hbar}$$

$$\Rightarrow i\hbar \partial_t \hat{O}(t) = [\hat{O}(t), \hat{H}]$$

let's calculate $\hat{\varphi}_{\alpha}(x,t)$:

$$i\hbar \partial_t \hat{\varphi}_{\alpha}(x,t) = [\hat{\varphi}_{\alpha}(x,t), \hat{H}]$$

$$e^{i\hat{H}t/\hbar}\, \hat{H}\hat{\varphi}_{\alpha}(x)\, e^{-i\hat{H}t/\hbar}$$

$$[\hat{\varphi}_{\alpha}(x,t), \hat{H}] = \underbrace{\hat{\varphi}_{\alpha}(x,t)\, \hat{H}}_{1} - \underbrace{\hat{H}\, \hat{\varphi}_{\alpha}(x,t)}_{2}$$

the time dependent phase factors in $\hat{\varphi}(x,t)$ commute with $\hat{H}$, hence:

$$\hat{H}\hat{\varphi}_{\alpha}(x) = -\gamma B \sum_{\alpha_1\alpha_2} \int dx'\, \hat{\varphi}^{+}_{\alpha_1}(x')\, \sigma^{z}_{\alpha_1\alpha_2}\, \hat{\varphi}_{\alpha_2}(x')\, \hat{\varphi}_{\alpha}(x)$$

$$= \mp \gamma B \sum_{\alpha_1\alpha_2} \int dx'\, \hat{\varphi}^{+}_{\alpha_1}(x')\, \hat{\varphi}_{\alpha}(x)\, \sigma^{z}_{\alpha_1\alpha_2}\, \hat{\varphi}_{\alpha_2}(x') \ominus$$

$$\hat{\varphi}^{+}_{\alpha_1}(x')\, \hat{\varphi}_{\alpha}(x) = \mp \delta_{\alpha_1\alpha_2}\, \delta(x-x') \pm \varphi_{\alpha}(x)\, \varphi^{+}_{\alpha_1}(x')$$

$$\ominus \gamma B \sum_{\alpha_1\alpha_2} \int dx' \left(\delta_{\alpha_1\alpha} \delta(x-x') - \hat{\varphi}_\alpha(x) \hat{\varphi}^+_{\alpha_1}(x') \right) \sigma^z_{\alpha_1\alpha_2} \varphi_{\alpha_2}(x')$$

overall we find that:

$$\hat{H} \hat{\varphi}_\alpha(x,t) = \gamma B \sum_{\alpha_2} \sigma^z_{\alpha\alpha_2} \varphi_{\alpha_2}(x,t) + \hat{\varphi}_\alpha(x,t) H$$

hence $[\hat{\varphi}_\alpha(x,t), \hat{H}] = -\gamma B \sum_{\alpha_2} \sigma^z_{\alpha\alpha_2} \varphi_{\alpha_2}(x,t)$

$$-\gamma B \sum_{\alpha_2} \sigma^z_{\alpha\alpha_2} \hat{\varphi}_{\alpha_2}(x,t) = \varepsilon_\alpha \hat{\varphi}_\alpha(x,t)$$

$$\text{where } \varepsilon_\alpha = \begin{cases} -\gamma B & \alpha = \uparrow \\ +\gamma B & \alpha = \downarrow \end{cases}$$

back to the Heisenberg EOM:

$$i\hbar \partial_t \hat{\varphi}_\alpha(x,t) = \varepsilon_\alpha \hat{\varphi}_\alpha(x,t)$$

$$\Rightarrow \hat{\varphi}(x,t) = \hat{\varphi}_\alpha(x,0) e^{-i\varepsilon_\alpha t/\hbar}$$

similar calculations (or just taking the hermitian conjugate of $\hat{\varphi}_\alpha$) lead to:

$$\hat{\varphi}^+_\alpha(x,t) = \hat{\varphi}^+_\alpha(x,0) e^{i\varepsilon_\alpha t/\hbar}$$

C.

$$i\hbar\,\partial_t \vec{S}(t) = [\vec{S}(t), H]$$

notice that $\hat{H} = \frac{-2\gamma B}{\hbar} S_z$

$$\Rightarrow i\hbar\,\partial_t S_j(t) = 2i\,\varepsilon_{zjk} S_k(t) \qquad j,k = x,y,z$$

$$\begin{cases} \partial_t S_x(t) = \frac{2\gamma B}{\hbar} S_y(t) \\ \partial_t S_y(t) = \frac{-2\gamma B}{\hbar} S_x(t) \\ \partial_t S_z(t) = 0 \end{cases}$$

we define the frequency $\omega = \frac{2\gamma B}{\hbar}$

$$S_x(t) = -\frac{1}{\omega}\partial_t S_y(t) \to \partial_t^2 S_y(t) = -\omega^2 S_y(t)$$

harmonic oscillator

general solution:

$$S_y(t) = C_1 \cos(\omega t) + C_2 \operatorname{sen}(\omega t)$$

$$S_x(t) = -\frac{1}{\omega}\partial_t S_y(t) = C_1 \operatorname{sen}(\omega t) - C_2 \cos(\omega t)$$

$$\begin{cases} S_y(t=0) = C_1 = S_y(0) \\ S_x(t=0) = -C_2 = S_x(0) \end{cases} \to \text{boundary conditions}$$

overall we find that:

$$\vec{S}(t) = \begin{cases} S_x(t) = S_x(0)\cos(\omega t) + S_y(0)\,\text{sen}(\omega t) \\ S_y(t) = S_y(0)\cos(\omega t) - S_x(0)\,\text{sen}(\omega t) \\ S_z(t) = S_z(0) \end{cases}$$

explicitly these components have the following form:

$$S_x(t) = \hbar/2 \int dx \left[\psi_\uparrow^\dagger \psi_\downarrow e^{-i\omega t} + \psi_\downarrow^\dagger \psi_\uparrow e^{i\omega t} \right]$$

$$S_y(t) = \hbar/2 \int dx \left[\psi_\uparrow^\dagger \psi_\downarrow (-i) e^{-i\omega t} + \psi_\downarrow^\dagger \psi_\uparrow (i) e^{i\omega t} \right]$$

$$S_z(t) = \hbar/2 \int dx \left[\psi_\uparrow^\dagger \psi_\uparrow - \psi_\downarrow^\dagger \psi_\downarrow \right]$$

d.

$$\vec{S}(t) = e^{iHt/\hbar}\,\hat{S}\,e^{-iHt/\hbar} =$$

$$= \frac{\hbar}{2}\sum_{\alpha\alpha'}\int dx\; e^{i\hat{H}t/\hbar}\,\hat{\psi}^+_\alpha(x,t)\,\vec{\sigma}_{\alpha\alpha'}\,\hat{\psi}_{\alpha'}(x,t)\,e^{-iHt/\hbar}$$

$$e^{-iHt/\hbar} \qquad e^{iHt/\hbar}$$

$$\vec{S}(t) = \hbar/2\sum_{\alpha\alpha'}\int dx\; \psi^+_\alpha(x,t)\,\vec{\sigma}_{\alpha\alpha'}\,\psi_{\alpha'}(x,t)$$

$$\begin{cases} \hat{\psi}_\alpha(x,t) = \psi_\alpha(x,0)\,e^{-i\varepsilon_\alpha t} \\ \hat{\psi}^+_\alpha(x,t) = \psi^+_\alpha(x,0)\,e^{i\varepsilon_\alpha t} \end{cases}$$

where ε_α was defined in point 6.

i) $$S_x(t) = \frac{\hbar}{2}\sum_{\alpha\alpha'}\int dx\; e^{i(\varepsilon_\alpha-\varepsilon_{\alpha'})t/\hbar}\,\hat{\psi}^+_\alpha(x)\,\sigma^x_{\alpha\alpha'}\,\hat{\psi}_{\alpha'}(x)$$

$$= \frac{\hbar}{2}\int dx\left[e^{-i\omega t}\hat{\psi}^+_\uparrow\hat{\psi}_\downarrow + e^{i\omega t}\hat{\psi}^+_\uparrow\hat{\psi}_\downarrow\right]$$

where we used $\varepsilon_{\uparrow,\downarrow} = \mp\gamma B$ and $\omega = 2\gamma B/\hbar$

ii) $$S_y(t) = \frac{\hbar}{2}\int dx\left[-i\,e^{-i\omega t}\hat{\psi}^+_\uparrow\hat{\psi}_\downarrow + i\,e^{i\omega t}\hat{\psi}^+_\downarrow\psi_\uparrow\right]$$

iii) $S_z(t) = \frac{\hbar}{2}\int dx \left[\psi_\uparrow^+(x)\psi_\downarrow(x) - \psi_\downarrow^+(x)\psi_\uparrow(x)\right] = S_z(0)$

hence the solutions obtained in b. and c. are consistent with each other

e.

$$|\psi_{\alpha_1\alpha_2}(t)\rangle = \psi^+_{\alpha_1}(x,t)\,\psi^+_{\alpha_2}(x,t)|0\rangle$$

$$\varepsilon_{\alpha_1\alpha_2} = \langle\psi_{\alpha_1\alpha_2}(t)|\hat{H}|\psi_{\alpha_1\alpha_2}(t)\rangle = \langle\psi_{\alpha_1\alpha_2}|\hat{H}|\psi_{\alpha_1\alpha_2}\rangle$$

$$= -\gamma B\langle 0|\psi_{\alpha_2}(x)\psi_{\alpha_1}(x)\sum_{\alpha\alpha'}\int dx'\,\psi^+_\alpha(x')\sigma^z_{\alpha\alpha'}\psi_{\alpha'}(x')\psi^+_{\alpha_1}(x)\psi^+_{\alpha_2}(x)|0\rangle$$

$$= -\gamma B\langle 0|\psi_{\alpha_2}(x)\psi_{\alpha_1}(x)\sum_{\alpha\alpha'}\int dx'\,\psi^+_\alpha(x')\sigma^z_{\alpha\alpha'}\left(\delta_{\alpha'\alpha_1}\delta(x-x') \pm \psi^+_{\alpha_1}(x)\psi_{\alpha'}(x')\right)\dots$$

$$\dots\psi^+_{\alpha_2}(x)|0\rangle =$$

$$= -\gamma B\langle 0|\psi_{\alpha_2}(x)\psi_{\alpha_1}(x)\left(\sum_\alpha\psi^+_\alpha(x)\sigma^z_{\alpha\alpha_1}\psi^+_{\alpha_2}(x) \pm \sum_{\alpha\alpha'}\int dx'\,\psi^+_\alpha(x)\sigma^z_{\alpha\alpha'}\psi_{\alpha_1}(x')\dots\right.$$

$\psi^+_{\alpha_2}\psi^+_{\alpha_2} = 0$

$$\dots\psi_{\alpha'}(x')\,\psi^+_{\alpha_2}(x)|0\rangle$$

$$= -\gamma B\,\sigma^z_{\alpha_1\alpha_1} \mp \gamma B\langle 0|\psi_{\alpha_2}(x)\psi_{\alpha_1}(x)\sum_{\alpha\alpha'}\int dx'\,\psi^+_\alpha(x')\sigma^z_{\alpha\alpha'}\psi^+_{\alpha_1}(x)\dots$$

$$\dots\left(\delta_{\alpha'\alpha_2}\delta(x-x') \pm \psi^+_{\alpha_2}(x)\psi_{\alpha'}(x')\right)|0\rangle =$$

$\hookrightarrow \psi_{\alpha'}(x')|0\rangle = 0$

$$= -\gamma B \sigma^z_{\alpha_1\alpha_1} \mp \gamma B \langle 0| \psi_{\alpha_2}(x) \psi_{\alpha_1}(x) \sum_\alpha \psi^+_\alpha(x) \sigma^z_{\alpha\alpha_2} \psi^+_{\alpha_1}(x)|0\rangle =$$

$$= -\gamma B \sigma^z_{\alpha_1\alpha_1} \mp \gamma B \langle 0| \psi_{\alpha_2}(x) \psi_{\alpha_1}(x) \psi^+_{\alpha_2} \psi^+_{\alpha_1} |0\rangle \sigma^z_{\alpha_2\alpha_2}$$

yields a minus sign for fermions

$$\Rightarrow \boxed{\mathcal{E}_{\alpha_1\alpha_2} = -\gamma B (\sigma^z_{\alpha_1\alpha_1} + \sigma^z_{\alpha_2\alpha_2})}$$

fermions

$\alpha_1 \neq \alpha_2$ Pauli exclusion principle

$\Rightarrow \mathcal{E}_{\uparrow\downarrow} = \mathcal{E}_{\downarrow\uparrow} = 0$ since $\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

bosons

$\alpha_1 = \alpha_2 = \uparrow$ $\qquad \mathcal{E}_{\uparrow\uparrow} = -2\gamma B$

$\alpha_1 = \alpha_2 = \downarrow$ $\qquad \mathcal{E}_{\downarrow\downarrow} = 2\gamma B$

$\alpha_1 \neq \alpha_2$ $\qquad \mathcal{E}_{\uparrow\downarrow} = \mathcal{E}_{\downarrow\uparrow} = 0$