Departement Natuur- en Sterrenkunde
Universiteit Utrecht

FINAL EXAM Quantum Field Theory - NS-TP401M

Thursday, February 1, 2018, 17:00-20:00, Educatorium Theatron.

1) Start every exercise on a separate sheet. Write on each sheet: your name and initials, and your student number.

2) Please write legibly and clear. Unreadable handwriting cannot be marked!

3) The exam consists of **four** exercises of indicated weight.

4) No lectures notes or any other material (books, calculators, ...) are allowed.

Formularium

In this exam we use natural units, in which $c = \hbar = 1$. The Minkowski metric in four spacetime dimensions is $\eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1)$. The Dirac gamma matrices are denoted by γ^μ and satisfy the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu} I$. Fourier transformation in d dimensions is given by

$$f(x) = \int \frac{\mathrm{d}^d k}{(2\pi)^d} e^{-ik\cdot x} \tilde{f}(k) \, , \qquad \tilde{f}(k) = \int \mathrm{d}^d x \, e^{ik\cdot x} f(x) \, . \tag{1}$$

Problem 1. [26 points]

This problem is comprised of short questions to test your general knowledge. You do not need to show any calculations in this problem, just write down the answer.

i.) (3P) Consider a Lorentz invariant classical field theory with a given action S. How does the Lagrangian density transform under a Lorentz transformation and how is this related to the invariance of S?

ii.) (4P) How does a Dirac field $\Psi_a(x)$ transform under a Lorentz transformation with boost and rotation parameters $\omega_{\mu\nu}$? And a real 4-vector field $A_\mu(x)$?

iii.) (3P) Is the Dirac representation reducible or irreducible?

iv.) (4P) Write down the action for a Dirac field and discuss how it is related to left- and right-handed Weyl fields by evaluating the kinetic term and the mass term. How many degrees of freedom does a Dirac field have?

v.) (4P) Consider the representation denoted by $(1, 1/2)$. What are the spin states contained in this representation? Are they bosons or fermions? Is this representation chiral or non-chiral (i.e. containing equal or different number of left and right handed particles).

vi.) (4P) What is the most general irreducible finite dimensional representation of Lorentz symmetry? What are the spin states contained in this representation? These finite representations are not unitary. How do we deal with this problem in quantum field theory?

vii.) (4P) What are the conserved quantities corresponding to invariance under space-time translations? How do these quantities transform under Lorentz transformations?

viii.) **[Bonus]**:(3P) Suppose that the action of the *classical* field is invariant under a particular global symmetry. What else should be required to maintain the same symmetry in the quantum theory?

Problem 2. [29 points]

i) (3P) Let $\mathcal{O}(\vec{x})$ be an operator defined in the Schrödinger picture. Given a Hamiltonian H, give the expression for the Heisenberg picture of this operator.

ii) (4P) Show that in this picture

$$i\frac{\partial}{\partial t}\mathcal{O} = [\mathcal{O}, H]. \tag{2}$$

Consider the following Lagrangian density for a complex scalar field ϕ with mass m

$$\mathcal{L} = -|\partial_\mu \phi|^2 - m^2|\phi|^2 - \frac{\lambda}{2}|\phi|^4, \tag{3}$$

where by the notation $(x_\mu)^2$ we mean $x_\mu x^\mu$. You will treat ϕ and $\phi^\dagger$ as independent fields below.

iii) (3P) Derive the equation of motion for ϕ for $\lambda = 0$.

iv) (3P) Express the general (Lorentz invariant) solution of this equation of motion in terms of the annihilation and creation operators $a_{\vec{k}}$, $a^\dagger_{\vec{k}}$ of a particle and $b_{\vec{k}}$, $b^\dagger_{\vec{k}}$ of an anti-particle with momentum $\vec{k}$, which satisfy the commutation relations $[a_{\vec{k}}, a^\dagger_{\vec{p}}] = [b_{\vec{k}}, b^\dagger_{\vec{p}}] = (2\pi)^3\delta^3(\vec{k} - \vec{p})$ and the rest vanishing.

v) (2P) Compute the generalized momenta π and $\pi^\dagger$ corresponding to ϕ and $\phi^\dagger$ and express the Hamiltonian density in terms of them.

vi) (3P) Compute the equation of motion of ϕ, again for $\lambda = 0$ using ii) and v) and check that it is the same as your result in iii). **[Hint:]** You can just use the commutation relation between canonical fields ϕ, $\phi^\dagger$ and momenta π, $\pi^\dagger$, no need to work out commutators between a's and b's.

vii) (3P) What global continuous symmetry does the lagrangian (3) has?

viii) (2P) Show that the following charge is conserved

$$Q = \frac{i}{2}\int d^3\vec{x}\,(\phi^\dagger\pi^\dagger - \phi\pi)\ .$$

ix) **[Bonus]** (3P) Express Q in terms of creation and annihilation operators and calculate the charge of each particle.

x) (3P) Write down the Feynman rules (in momentum space) for the propagator and the vertex for this theory.

xi) (3P) Draw the Feynman diagrams contributing to the scattering amplitude for the process $\phi\phi^\dagger \to \phi\phi^\dagger$, up to order λ^2. Do not evaluate them.

xii) **[Bonus]** (2P) Give the symmetry factors of these diagrams.

Problem 3. [30 points]

Reconsider the same theory in (3) but this time coupled to a gauge field. Adding the kinetic term for the gauge field, the Lagrangian for our theory becomes

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 - |D_\mu\phi|^2 - m^2|\phi|^2 - \frac{\lambda}{2}|\phi|^4\,, \tag{4}$$

where $D_\mu = \partial_\mu - ieA_\mu$.

i) (3P) This lagrangian has a *local* symmetry now. How do ϕ, $\phi^\dagger$ and A_μ transform under this local symmetry?

ii) (7P) Write down the Feynman rules (in momentum space) for the new vertices (those involving A_μ) and draw the diagrams contributing to the gauge field propagator at order e^2 (these are one-loop diagrams). Do not evaluate the diagrams.

iii) (20P) We now wish to compute the scattering amplitude for the scattering process $\phi\phi^\dagger \to \phi\phi^\dagger$, at order e^2. The LSZ reduction formula for this process, with the incoming momenta labeled k_1 and k_2 and the outgoing momenta k_1' and k_2',

$$\begin{aligned}\langle f|i\rangle = \int d^4x_1 e^{ik_1x_1}(-\partial_1^2+m^2)\ d^4x_2 e^{ik_2x_2}(-\partial_2^2+m^2)\\ d^4x_1' e^{-ik_1'x_1'}(-\partial_{1'}^2+m^2)\ d^4x_2' e^{-ik_2'x_2'}(-\partial_{2'}^2+m^2)\\ \times \langle 0|\, T\phi^\dagger(x_1)\phi(x_2)\phi(x_1')\phi^\dagger(x_2')\,|0\rangle\,,\end{aligned} \tag{5}$$

brings the scattering amplitude in the following form

$$\langle f|i\rangle = (2\pi)^4\,\delta^4(k_1+k_2-k_1'-k_2')\,i\mathcal{T}, \tag{6}$$

where $i\mathcal{T}$ is the sum of the diagrams contributing to this process, with the propagators of the external scalar fields replaced by 1. Explain in one line why this simplification occurs? Now draw all the diagrams that contribute, and compute $\mathcal{T}$. For this, you will need the propagator of the vector field

$$\Pi^{\mu\nu} = -\frac{i}{k^2 - i\epsilon}\left(\eta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}(1-\xi)\right). \tag{7}$$

Do not work in a specific gauge, i.e. leave the parameter ξ arbitrary. Express you result in terms of the Mandelstam variables

$$s \equiv -(k_1 + k_2)^2 = -(k_1 + k_2)^2 \tag{8}$$

$$t \equiv -(k_1 - k_1')^2 = -(k_2 - k_2')^2 \tag{9}$$

$$u \equiv -(k_1 - k_2')^2 = -(k_2 - k_1')^2. \tag{10}$$

Does your result depend on ξ? Was this expected?

Problem 4. [15 points]

Now consider making the replacement $m \to i\mu$ in the Lagrangian (4). One obtains the following Lagrangian

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 - |D_\mu\phi|^2 - V(\phi) \tag{11}$$

where the potential is

$$V(\phi) = -\mu^2|\phi|^2 + \frac{\lambda}{2}|\phi|^4 \tag{12}$$

with μ and λ positive.

i) (8P) Sketch this potential as a function of $\phi_1 = \mathrm{Re}\,\phi$ and $\phi_2 = \mathrm{Im}\,\phi$, as a 3D plot. This potential has a continuum of minima, which all have $|\phi_{\mathrm{min}}| = \phi_0$. Find ϕ_0 and explain why all the minima have the same norm. Now one can expand $V(\phi)$ around the minimum ϕ_0, as $\phi(x) = \phi_0 + \frac{1}{\sqrt{2}}(\delta\phi_1 + i\delta\phi_2)$ for $\delta\phi_1, \delta\phi_2 \ll \phi_0$ (this means that ϕ acquires a vacuum expectation value $\langle\phi\rangle = \phi_0$ and the $U(1)$ global symmetry is "spontaneously broken"). What do you expect for the masses of $\delta\phi_1$ and $\delta\phi_2$ to be — positive, negative or null?

ii) (7P) Expand the kinetic term $|D\phi|^2$, neglecting terms cubic and quartic in A, $\delta\phi_1$ and $\delta\phi_2$; Identify the coupling between A_μ and $\delta\phi_2$, as well as the mass m_A that the gauge field A_μ has acquired. (This process of spontaneously breaking a symmetry and generating a mass for the gauge field is known as the Higgs mechanism.)

iii) **[Bonus]**: (3P) compute the propagotar for such a massive vector field

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 - \frac{1}{2}m_A^2 A_\mu A^\mu - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 \tag{13}$$

$$= \frac{1}{2}A_\mu\left(\eta^{\mu\nu}\partial^2 - \left(1 - \frac{1}{\xi}\right)\partial^\mu\partial^\nu - m_A^2\eta^{\mu\nu}\right)A_\nu \tag{14}$$

where we reintroduced the gauge fixing parameter (which has to be included since the mass term of the vector field is spontaneously generated).