Institute for Theoretical Physics, Faculty of Physics and Astronomy, UU.
Made available in electronic form by the $\mathcal{TBC}$ of A−Eskwadraat
In 2005/2006, the course SFT was given by Prof. Dr. Stoof.

Statistical Field Theory Midterm take home (SFT)
29 November 2005

Question 1. Hubbard-Stratonovich transformation

Consider an interacting gas N spinless fermions on a one-dimensional line with length L. The action for this gas at temperature $T = 1/k_B\beta$ and chemical potential μ is given by

$$S[\phi^*, \phi] = \int_0^{\hbar\beta} d\tau \int dx \phi^*(x,\tau) \left(\hbar \frac{\partial}{\partial \tau} - \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} - \mu \right) \phi(x,\tau) + \tag{1}$$

$$\frac{1}{2} \int_0^{\hbar\beta} d\tau \int dx \int dx' \phi^*(x,\tau)\phi^*(x',\tau) V(x-x') \phi(x',\tau)\phi(x,\tau) \tag{2}$$

where m is the mass of the fermions and $X(x-x')$ the interaction potential. We want to perform a Hubbard-Stratonovich transformation to the field $n(x,x',\tau)$ that on average is equal to the density matrix of the Fermi gas, so $\langle n(x,x',\tau) \rangle = \langle \phi^*(x,\tau)\phi(x',\tau) \rangle$. To still be able to calculate the exact fermionic Green's function $G(x,\tau;x',\tau')$, even after the Hubbard-Stratonovich transformation and integrating out the fermionic fields, we add also the current terms

$$S[\phi^*, \phi, J] = -\hbar \int_0^{\hbar\beta} d\tau \int dx (\phi^*(x,\tau) J(x,\tau) + J^*(x,\tau)\phi(x,\tau))$$

to the action. In the following we denote the fermionic Green's function with the Fock-like selfenergy $\hbar\Sigma(x,\tau;x',\tau') = \delta(\tau - \tau') X(x-x') n(x',x,\tau)$ by $G(x,\tau;x',\tau';n)$.

a) Perform the usual Hubbard-Stratonovich transformation that decouples the interaction between the fermions and integrate out the fermions. Determine the effective action $S^{eff}[n;J,J^*] \equiv S^{eff}[n] + S^{eff}[J,J^*]$. Show that in particular that

$$S^{eff}[J,J^*] = \hbar \int_0^{\hbar\beta} d\tau \int dx \int_0^{\hbar\beta} d\tau' \int \int dx' J^*(x,\tau) G(x,\tau;x',\tau';n) J(x',\tau').$$

b) Prove now the the exact fermionic Green's function can be obtained from

$$G(x,\tau;x'\tau') = \langle G(x,\tau;x',\tau';n) \rangle$$

where the average in the left hand side denotes the averega over the field $n(x,x',\tau)$ using the effective $S^{eff}[n]$.

It is possible to obtain also all the correlation functions of the product $\phi^*(x,\tau)\phi(x,\tau)$ in the above manner, but in this case it is easier to perform the calculation differently. Instead of adding the current term $S[\phi^*,\phi;J,J^*]$, we now add the source term

$$S[\phi^*\phi;K] = -\hbar \int_0^{\hbar\beta} d\tau \int dx \int dx' \phi^*(x,\tau)\phi(x',\tau) K x', x, \tau)$$

to the action $S[\phi^*,\phi]$. We can assume that $K^*(x'x,x',\tau) = K(x,x',\tau)$.

c) Perform a Hubbard-Stratonovich transformation that simultaneously decouples the interaction between the fermions and also removes the above source term from the fermionc part of the action. Integrate out the fermions and calculate the effective action $S^{eff}[n;K]$.

d) Prove from the last result that

$$\langle \phi^*(x,\tau)\phi(x',\tau)\rangle = \langle n(x,x',\tau)\rangle,$$

and

$$\begin{aligned}\langle \phi^*(x,\tau)\phi(x',\tau)\phi^*(x'',\tau')\phi(x''',\tau')\rangle &= \langle n(x,x',\tau)n(x'',x''',\tau')\rangle \\ &- \hbar V^{-1}(x',x,x'',x''')\delta(\tau-\tau'),\end{aligned}$$

with $V(x,x',x'',x''') \equiv V(x-x')\delta(x-x'')\delta(x'-x''')$. If you have not calculated the effective action $S^{eff}[n;K]$ in part c), deduce from the above relations how it should look like.