Institute for Theoretical Physics, Faculty of Physics and Astronomy, UU.
Made available in electronic form by the $\mathcal{TBC}$ of A−Eskwadraat
In 2004/2005, the course NS-TP402M was given by Prof. dr. Stoof.

Statistical Field Theory (NS-TP402M)
14 February 2005

Question 1. Coupled bosons and fermions

In this exercise we consider a system of coupled bosonic molecules and fermionic atoms, in a volume V and described by the fields $\phi_{\rm B}(\mathbf{x},\tau)$ and $\phi_{\rm F}(\mathbf{x},\tau)$, respectively. The action for this system is

$$S[\phi_{\rm B}^*,\phi_{\rm B},\phi_{\rm F}^*,\phi_{\rm F}] = \int_0^{\hbar\beta} d\tau \int d\mathbf{x} \phi_{\rm B}^*(\mathbf{x},\tau)\left(\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2}{2m_{\rm B}}\frac{\partial^2}{\partial\mathbf{x}^2} + \delta - 2\mu\right)\phi_{\rm B}(\mathbf{x},\tau)$$
$$+ \sum_{\sigma=\uparrow,\downarrow}\int_0^{\hbar\beta} d\tau \int d\mathbf{x}\phi_{{\rm F},\sigma}^*(\mathbf{x},\tau)\left(\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2}{2m_{\rm F}}\frac{\partial^2}{\partial\mathbf{x}^2} - \mu\right)\phi_{{\rm F},\sigma}(\mathbf{x},\tau)$$
$$\int_0^{\hbar\beta} d\tau \int d\mathbf{x}\; g\left[\phi_{\rm B}^*(\mathbf{x},\tau)\phi_{{\rm F},\uparrow}(\mathbf{x},\tau)\phi_{{\rm F},\downarrow}(\mathbf{x},\tau) + \phi_{\rm B}(\mathbf{x},\tau)\phi_{{\rm F},\downarrow}^*(\mathbf{x},\tau)\phi_{{\rm F},\uparrow}^*(\mathbf{x},\tau)\right], \qquad (1)$$

where $m_{\rm B} = 2m_{\rm F}$ is the mass of the molecules, $m_{\rm F}$ is the mass of the atoms, δ is the so-called detuning, μ is the chemical potential, and g is the strength of the atom-molecule coupling.

a) Give a brief description of the physical meaning of the term that couples the bosons and fermions in the above action. Do you understand why we need two different spin states for the fermions? Which term in the above action controls the energy difference between the zero-momentum states of the bosons and fermions?

b) First we want to consider the limit of very large positive detuning where the bosonic field has an expectation value equal to zero. In this limit we can integrate out the bosonic field directly. Show that by integrating out the bosons an interaction term between the fermions is generated. What is the strength of this interaction at zero momentum and zero Matsubara frequency?

Let us now assume that the bosons are Bose-Einstein condensed and, therefore, that the field $\phi_{\rm B}(\mathbf{x},\tau)$ has a nonzero expectation value $\langle\phi_{\rm B}({\rm x},\tau)\rangle \equiv \sqrt{n_0}$.

c) Replace the bosonic field by its expectation value and determine the resulting inverse Green's function matrix $\mathbf{G}^{-1}(\mathbf{k},i\omega_n)$ of the fermions by performing a Fourier transformation of both the space and time dependence.

d) Determine the total number of atoms in the gas, by making use of the Green's function $\mathbf{G}(\mathbf{k},i\omega_n)$. If you could not answer part c) give your answer in terms of $\mathbf{G}(\mathbf{k},i\omega_n)$. What is the total number of molecules?

e) Finally, the off-diagonal entry $\mathbf{G}_{12}(\mathbf{k},i\omega_n)$ corresponds to the expectation-value $\langle\phi_{{\rm F},\uparrow}(\mathbf{x},\tau)\phi_{{\rm F},\downarrow}(\mathbf{x},\tau)\rangle$. This value is related to the expectation value $\langle\phi_{\rm B}(\mathbf{x},\tau)\rangle$ of the Bose-Einstein condensed bosons. Determine this relation by minimizing the action with respect to $\phi_{\rm B}^*(\mathbf{x},\tau)$ and evaluate the Matsubara sum in the result obtained.

Question 2. Condensate in a harmonic trap

For very low temperatures the Bose-Einstein condensate wavefunction $\phi_0(\mathbf{x})$ is the solution of the Gross-Pitaevskii equation $\left(-\hbar^2\nabla^2/2m + V^{\mathrm{ex}}(\mathbf{x}) + V_0|\phi_0|^2\right)\phi_0(\mathbf{x}) = \mu\phi_0(\mathbf{x})$, where the condensate wavefunction $\phi_0(\mathbf{x})$ is normalized to the number of condensate atoms N_0, i.e., $\int d\mathbf{x}\ \phi_0^*(\mathbf{x}) = N_0$. The mass of the atoms is m and their interaction potential is $V_0\delta(\mathbf{x}-\mathbf{x}')$. This is a nonlinear equation, which is in general hard to solve analytically. However, there are certain limits, where analytic expressions can be obtained. We want to solve this equation for the case of the harmonic potential

$$V^{\mathrm{ex}}(\mathbf{x}) = \tfrac{1}{2}m\omega^2\left(x^2+y^2+z^2\right). \tag{2}$$

a) Suppose that the gas is weakly interacting, such that we can neglect the interaction term. Then the Gross-Pitaevskii equation reduces to the Schrödinger equation for a three-dimensional harmonic oscillator. Solve the Gross-Pitaevskii equation for this case and give the condensation wavefunction. What is the chemical potential μ? Can you understand your result? Does the condensate size depend on the number of condensate atoms? Give the physical reason for your answer.

b) For a weakly-interacting gas the interaction term can be treated in pertubation theory. Calculate the first-order correction to the chemical potential. What is the physical interpretation of this correction term? If you could not solve part a), denote the non-interacting condensate waveform by $\psi_0(\mathrm{x})$ and give the correction to μ in terms of $\psi_0(\mathrm{x})$.

c) When the gas is strongly interacting, the kinetic term $-\hbar^2\nabla^2/2m$ can be neglected. Solve the Gross-Pitaevskii equation in this approximation and prove that the condensate density can be written as

$$n_0(\mathbf{x}) = \phi_0^*(\mathbf{x})\phi_0(\mathbf{x}) = \begin{cases} \frac{15N_0}{8\pi R^3}\left(1-\frac{r^2}{R^2}\right) & \text{if } r \le R; \\ 0 & \text{if } r > R. \end{cases} \tag{3}$$

This is called the Thomas-Fermi profile and R is called the Thomas-Fermi radius.

d) Express the Thomas-Fermi radius R and the chemical potential μ in terms of the condensate atoms N_0 and the interaction strength V_0. What do you expect for the dependence of R and μ as a function of N_0?

e) Give an estimate for the magnitude of the kinetic term in the Gross-Pitaevskii equation within the Thomas-Fermi approximation. This term should be small compared to the interaction term. Use this to prove that the Thomas Fermi approximation is valid if the number of condensate atoms obeys

$$N_0 \gg \frac{\hbar^2 R}{mV_0}. \tag{4}$$