Statistical Field Theory Exam

January 29th, 2008
Duration of the exam: 3 hours

1. Use a separate sheet for every exercise.

2. Write your name and initials in all sheets, on the first sheet also your student ID number.

3. Write clearly, unreadable work cannot be corrected.

4. You are NOT allowed to use any kind of books or lecture notes.

Ising Model in 3d

Consider the 3-dimensional Ising model defined by the partition function

$$\mathcal{Z} = \sum_{\{S_i\}} e^{\sum_{ij} S_i K_{ij} S_j + \sum_i h_i S_i}, \tag{1}$$

where the correlation matrix $K_{ij} := \beta C\left(|i-j|\right)$ accounts for the mutual spin interactions, $h_i := \beta H_i$, with H_i an external magnetic field, $S_i \in \{1, -1\}$, and the sums in the exponent run over the 3-d lattice sites.

(1.0) **1.** We want to compute the partition function for this model approximately. To this end, let us first note that

$$1 = \mathcal{N} \int \mathcal{D}[\psi] e^{-\frac{1}{4}\sum_{ij} \psi_i K^{-1}_{ij} \psi_j},$$

with $\mathcal{D}[\psi] := \prod_i d\psi_i$, K^{-1} the inverse of the correlation matrix and $\mathcal{N} = \sqrt{\det\left(4\pi K^{-1}\right)}$ a normalization factor. Do a shift of the integration variables ψ_i in the expression above to get

$$1 = \mathcal{N} \int \mathcal{D}[\psi] e^{-\frac{1}{4}\sum_{ij} \psi_i K^{-1}_{ij} \psi_j + \sum_i S_i \psi_i - \sum_{ij} S_i K_{ij} S_j}.$$

(1.0) **2.** Now, by performing a Hubbard-Stratonovich transformation, show that

$$\mathcal{Z} = \mathcal{N} \int \mathcal{D}[\psi] \sum_{\{S_i\}} e^{-\frac{1}{4}\sum_{ij} \psi_i K^{-1}_{ij} \psi_j + \sum_i S_i(\psi_i + h_i)}.$$

(1.0) **3.** Finally, absorbing any eventual inessential factors in the normalization term, calculate the summation $\sum_{\{S_i\}}$ in the partition function to get

$$\mathcal{Z} = \mathcal{N} \int \mathcal{D}[\psi] e^{-\frac{1}{4}\sum_{ij} (\psi_i - h_i) K^{-1}_{ij} (\psi_j - h_j) + \sum_i \ln(2\cosh\psi_i)},$$

and switch to integration variables $\phi_i := \frac{1}{2}\sum_j K_{ij}^{-1}\psi_j$ to arrive at

$$\mathcal{Z} = \mathcal{N}\int \mathcal{D}[\phi] e^{-\sum_{ij}\phi_i K_{ij}\phi_j + \sum_i \phi_i h_i + \sum_i \ln\cosh\left(2\sum_j K_{ij}\phi_j\right) + \mathcal{O}(h^2)} .$$

(1.0) **4.** We now resort to the simplifying assumption that we are working at low temperatures, so that $|\phi_i| \ll 1$. Then, performing a Fourier transformation $\phi_i \to \phi_{\mathbf{k}}$, $K_{ij} \to K_{\mathbf{k}}$, using the expansion $\ln\cosh(x) = \frac{1}{2}x^2 - \frac{1}{12}x^4 + \ldots$, and noting $(K\phi)(\mathbf{k}) = K(\mathbf{k})\phi(\mathbf{k}) = K(0)\phi(\mathbf{k}) + \frac{1}{2}\mathbf{k}^2 K''(0)\phi(\mathbf{k}) + \mathcal{O}(\mathbf{k}^4)$, show that

$$\begin{aligned} S[\phi] = & \sum_{\mathbf{k}} \left[\phi_{\mathbf{k}}\left(c_1 + c_2 \mathbf{k}\cdot\mathbf{k}\right)\phi_{-\mathbf{k}} + c_3 \phi_{\mathbf{k}} h_{-\mathbf{k}}\right] \\ & + \frac{c_4}{N}\sum_{\mathbf{k}_1,\ldots,\mathbf{k}_4} \phi_{\mathbf{k}_1}\phi_{\mathbf{k}_2}\phi_{\mathbf{k}_3}\phi_{\mathbf{k}_4}\delta_{\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4,0} + \mathcal{O}(\mathbf{k}^4, h^2, \phi^6, \mathbf{k}^2\phi^4), \end{aligned}$$

where the coefficients c_i are given by

$$c_1 = K(0)\left(1 - 2K(0)\right), \quad c_2 = \tfrac{1}{2}K''(0)\left(1 - 4K(0)\right), \quad c_3 = 1 \quad c_4 = \tfrac{4}{3}K(0)^4 .$$

(1.0) **5.** In the continuum limit, $S[\phi]$ takes the form

$$S[\phi] = \int d^3x \left[c_2 (\partial\phi)^2 + c_1\phi^2 + c_3\phi h + N c_4 \phi^4\right] .$$

Is there a phase transition in this system? If so, determine the critical temperature T_c by analyzing the coefficients of the action above and discuss the order of this phase transition.

Superfluid-Mott Insulator transition

One of the most important experiments of the past decade in the field of cold atoms was the first observation of a true quantum phase transition. The transition was observed with bosonic atoms in a so-called optical lattice, which is a periodic potential for the bosons created by laser light. The transition was from a gapless superfluid phase in which the atoms are free to move trough the lattice to a gapped so-called Mott insulator phase, in which there is precisely one atom per site. The system is described by a Bose-Hubbard Hamiltonian and the action that we will use to study the system is

$$\begin{aligned} S[a^*, a] \;=\; & \int_0^{\hbar\beta} d\tau \left[\sum_i a_i^*(\tau)\left(\hbar\frac{\partial}{\partial\tau} - \mu\right)a_i(\tau) - \sum_{i,j} t_{ij} a_i^*(\tau) a_j(\tau)\right. \\ & \left. + \frac{1}{2}U\sum_i a_i^*(\tau)a_i^*(\tau)a_i(\tau)a_i(\tau)\right], \end{aligned}$$

where $a_i^{(*)}$ is the field corresponding to annihilation (creation) of a boson at optical lattice site i, μ is the chemical potential, U is the on-site interaction energy, t_{ij} are the tunneling coefficients (t for nearest neighbours, 0 otherwise) and the summations are over all lattice sites.

(1.0) **1.** Perform a Hubbard-Stratonovich transformation such that you obtain an effective action for the atoms that has the form

$$\begin{aligned} S_{\text{eff}}[a^*, a, \psi^*, \psi] &= \int_0^{\hbar\beta} d\tau \left[\sum_i a_i^*(\tau) \left(\hbar \frac{\partial}{\partial \tau} - \mu \right) a_i(\tau) + \frac{1}{2} U \sum_i a_i^*(\tau) a_i^*(\tau) a_i(\tau) a_i(\tau) \right. \\ &\left. - \sum_{i,j} t_{ij} \left(a_i^*(\tau)\psi_j(\tau) + \psi_i^*(\tau) a_j(\tau) \right) + \sum_{i,j} t_{ij} \psi_i^*(\tau)\psi_j(\tau) \right]. \end{aligned}$$

We define $S^0_{\text{eff}}[a^*, a]$ as $S_{\text{eff}}[a^*, a, \psi^*, \psi]$ with $t_{ij} = 0$. Furthermore, we define the average

$$\langle A[a^*, a] \rangle_0 \equiv \int d[a^*] d[a] A[a^*, a] \exp\left\{ -\frac{1}{\hbar} S^0_{\text{eff}}[a^*, a] \right\}.$$

(1.0) **2.** Show, with the above definitions, that up to quadratic order in the fields ψ the effective action for the non-condensed phase $S^{(2)}_{\text{eff}}[\psi^*, \psi]$ becomes

$$\begin{aligned} S^{(2)}_{\text{eff}}[\psi^*, \psi] &= -\frac{1}{2\hbar} \left\langle \left(\int_0^{\hbar\beta} d\tau \sum_{i,j} t_{ij} (a_i^*(\tau)\psi_j(\tau) + \psi_i^*(\tau) a_j(\tau)) \right)^2 \right\rangle_0 \\ &+ \int_0^{\hbar\beta} d\tau \sum_{i,j} t_{ij} \psi_i^*(\tau)\psi_j(\tau). \end{aligned}$$

Hints: 1) For the non-condensed case $\langle a_i \rangle = 0$. 2) To calculate expectation values, expand the exponential. To get the correct term for the effective action, re-exponentiate.

It turns out that one can actually calculate the above expectation values using perturbation theory (you don't have to do it!). Transforming the fields to Fourier space, we finally obtain

$$S^{(2)}_{\text{eff}}[\psi^*, \psi] = -\sum_{\mathbf{k},n} |\psi_{\mathbf{k}n}|^2 \epsilon_{\mathbf{k}} \left[1 + \epsilon_{\mathbf{k}} \left(\frac{1+g}{-i\hbar\omega_n - \mu + gU} + \frac{g}{i\hbar\omega_n + \mu - (g-1)U} \right) \right],$$

where g is the number of particles per site, $\epsilon_{\mathbf{k}} = -2t \sum_{j=1}^{d} \cos(k_j l)$ with d the dimensionality of the lattice (usually 3) and l the lattice spacing.

(1.0) **3.** By performing the substitution $i\omega_n \to \omega$, and putting in the above expression the term between square brackets equal to zero, you could in principle obtain the quasi-particle/hole dispersions (don't calculate it). Why is this the case? What are you putting to zero?

It turns out that the dispersion for the density fluctuations, given by the difference between the quasi-particle dispersion and the quasi-hole dispersion, is given by

$$\hbar\omega_{qp} - \hbar\omega_{qh} = \Delta E_{\mathbf{k}} = \sqrt{\epsilon_{\mathbf{k}}^2 + (4g+2)U\epsilon_{\mathbf{k}} + U^2}.$$

(1.0) **4.** Take one particle per site, which means g equal to 1. Note that t is positive. For which $\mathbf{k}$ do we have a minimum of the above dispersion? Remember that the Mott-insulator has a gapped dispersion (and precisely one particle per site). Is the system in the Mott-insulator phase for very large and positive U? Can you understand the gapped spectrum physically? Show that the transition to the gapless superfluid occurs when $U/Zt = 3 + 2\sqrt{2} \approx 5.8$, where $Z = 2d$ is the number of nearest neighbours.

(1.0) **5.** Consider now a generalized Bose-Hubbard model, with anisotropic and complex hopping coefficients,

$$\hat{H} = -\sum_{\mathbf{r}\in A, j=1-4} \left\{ |c| e^{i\theta(-1)^j} \hat{a}_{\mathbf{r}}^\dagger \, \hat{a}_{\mathbf{r}+\mathbf{e}_j} + \text{H.c.} \right\} + \frac{1}{2} U \sum_{\mathbf{r}\in A\oplus B} \hat{n}_{\mathbf{r}} \left(\hat{n}_{\mathbf{r}} - 1\right),$$

where A and B refer to a bipartite lattice, see Fig. (1).

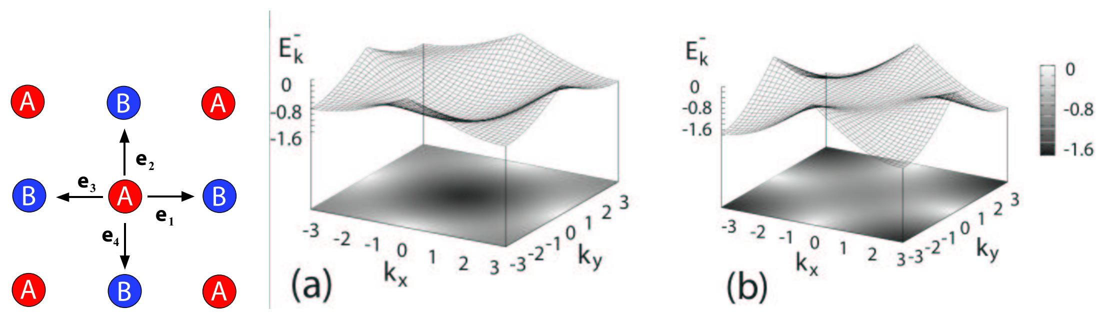

Figure 1: The bipartite lattice comprises sites labeled by "A" and "B". The grey rectangle identifies a single plaquette. Single-particle spectrum for (a) $\theta < \pi/4$, with the minimum at the origin, and (b) $\theta > \pi/4$, with minima at the Brillouin zone edges $(\pm\pi, \pm\pi)$.

The single-particle spectrum of this problem reads

$$E_{\mathbf{k}}^{\pm} = \pm|\epsilon_{\mathbf{k}}| = \pm 2|c| \left[\cos^2\left(k^+ l\right) + \cos^2\left(k^- l\right) + 2\cos\left(k^+ l\right)\cos\left(k^- l\right)\cos\left(2\theta\right) \right]^{1/2},$$

where $k^{\pm} = (k_x \pm k_y)/2$ and l is the lattice constant. The spectrum has two branches, but since we are interested on bosons, we consider only the lowest one, which is shown in Fig. 1(a) and (b). We observe that for $\theta < \pi/4$ the spectrum has a minimum at $\mathbf{k} = (k_x, k_y) = (0,0) = 0$, whereas for $\theta > \pi/4$ the minima are at $\mathbf{k} = (k_x, k_y) = (\pm\pi, \pm\pi) = \pi$

(these four points are equivalent). The wave function describing the bosons is the same in all lattice sites for $\theta < \pi/4$, whereas its angular phase σ changes by $\pi/2$ between neighboring sites if $\theta > \pi/4$ (the wave function picks up a phase 2π while going around a plaquette, i.e., vortices and anti-vortices set in). At $\theta = \pi/4$ the minima at $\mathbf{k} = 0$ and $\mathbf{k} = \pi$ have equal energy. What kind of phase transition are we describing at $\theta = \pi/4$? How should the thermodynamical potential for this transition look like? What can you say about the coeficients of the Landau-free energy functional?