Midterm Exam - "Statistical Field Theory"

November 7, 2006
Duration of the exam: 3 hours

1. Use a separate sheet for every exercise.
2. Write your name and initials in all sheets, on the first sheet also your address and your student ID number.
3. Write clearly, unreadable work cannot be corrected.
4. You are NOT allowed to use any kind of books or lecture notes.

Exercise I - Landau-Ginzburg theory

Landau-Ginzburg theory is an effective way to describe fluctuations in, for example, the Ising model. The (effective) Landau free energy is given by

$$F_L[\phi] = \frac{1}{2}\int d\mathbf{r}\left\{K\Big(\nabla\phi(\mathbf{r})\Big)^2 + M^2\phi(\mathbf{r})^2\right\}, \tag{1}$$

where M and K are both positive real numbers and also the fluctuating fields $\phi(\mathbf{r})$ are real.

1) By introducing the following Fourier transformation

$$\phi(\mathbf{r}) = \frac{1}{\sqrt{V}}\sum_{\mathbf{q}} e^{-i\mathbf{qr}}\phi_{\mathbf{q}}, \tag{2}$$

show that F_L can be written as

$$F_L[\phi] = \frac{1}{2}\sum_{\mathbf{q}}\left\{M^2 + Kq^2\right\}\phi_{\mathbf{q}}\phi_{-\mathbf{q}}, \tag{3}$$

where we note that $\phi_{-\mathbf{q}} = \phi^*_{\mathbf{q}}$ since $\phi(\mathbf{r})$ is real.

The Landau free energy is related to the partition sum Z in momentum space in the following way

$$Z = \int[d\phi]e^{-\beta F_L[\phi]}, \tag{4}$$

with $\beta = 1/k_BT$. Since we have obtained a very convenient expression for F_L in momentum space, it turns out that you can calculate the correlations $G_{\mathbf{q},\mathbf{q}'} \equiv \langle\phi_{\mathbf{q}}\phi^*_{\mathbf{q}'}\rangle$ exactly by simply applying the Gaussian integration rules that we have learned during the lectures (you don't need to do it here!). This gives

$$G_{\mathbf{q},\mathbf{q}'} \equiv \langle\phi_{\mathbf{q}}\phi^*_{\mathbf{q}'}\rangle = \langle\phi_{\mathbf{q}}\phi_{-\mathbf{q}'}\rangle = \frac{1}{M^2+Kq^2}\delta_{\mathbf{q},\mathbf{q}'}. \tag{5}$$

2) Now, using the knowledge of the correlations in momentum space, you are going to determine the correlation function in real space $G(\mathbf{r})$. Show that

$$G(\mathbf{r}) \equiv \langle \phi(0)\phi(\mathbf{r})\rangle = \frac{1}{(2\pi)^3}\int d\mathbf{q}\frac{e^{i\mathbf{qr}}}{M^2 + Kq^2}. \tag{6}$$

Hint: Use $\frac{1}{V}\sum_{\mathbf{q}} \to \int d\mathbf{q}/(2\pi)^3$.

The momentum integral in eq. (6) can be solved exactly and yields

$$G(r) = \frac{1}{4\pi K}\frac{e^{-r/\xi}}{r}, \tag{7}$$

where $\xi = \sqrt{K}/M$ is the so-called correlation length. This you are going to show in two steps.

3) First, introduce spherical coordinates $\int d\mathbf{q} \to \int q^2 \sin(\theta)\, dq\, d\phi\, d\theta$ into eq. (6) and perform the resulting integrals over the angles. Show that these integrals result in

$$G(r) = \frac{1}{ir(2\pi)^2}\int_0^\infty dq \frac{q(e^{iqr} - e^{-iqr})}{M^2 + Kq^2}. \tag{8}$$

4) This last integral can conveniently be solved by using contour integration, since the integrant has two poles in the complex plane. In this way, show that (8) indeed leads to (7).

Exercise II - Bose-Fermion mixture and Gaussian integrals

Suppose we have a mixture of N fermionic and M bosonic degrees of freedom. The fermionic degrees of freedom are described by N complex Grassmann variables $\psi_i, i = 1 \ldots N$ and the bosons are described by M complex variables $\phi_i, i = 1 \ldots M$. One may combine these degrees of freedom in a single vector variable (vector of length $N + M$):

$$X = \begin{pmatrix} \Psi \\ \Phi \end{pmatrix},\ X^+ = (\Psi^+\, \Phi^+) \tag{9}$$

One can say that the vector X is neither "commuting" nor "anticommuting" since it contains elements of both types. Components of X are the following:

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \ldots \\ \psi_N \end{pmatrix},\ \Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \ldots \\ \phi_M \end{pmatrix},\ \Psi^+ = (\psi_1^* \ldots \psi_N^*),\ \Phi^+ = (\phi_1^* \ldots \phi_M^*) \tag{10}$$

One may define the generalized matrix consisting of both commuting (usual complex numbers) and Grassmann anticommuting elements:

$$G = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{11}$$

where A, D are correspondingly $N \times N$ and $M \times M$ square matrices, elements of which are usual complex commuting numbers; B, C are, correspondingly, $N \times M$ and $M \times N$ matrices, which consist of Grassmann anticommuting numbers:

$$B = \begin{pmatrix} B_{11} & B_{12} & \dots & \dots & B_{1M} \\ B_{21} & B_{22} & \dots & \dots & B_{2M} \\ \dots & \dots & \dots & \dots & \dots \\ B_{N1} & B_{N2} & \dots & \dots & B_{NM} \end{pmatrix}, \; C = \begin{pmatrix} C_{11} & C_{12} & \dots & C_{1N} \\ C_{21} & C_{22} & \dots & C_{2N} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ C_{M1} & C_{M2} & \dots & C_{MN} \end{pmatrix} \tag{12}$$

Now one can construct a general discretized form of an "action" for this system

$$S(X, X^+) = X^+ G X \tag{13}$$

and calculate the corresponding "partition sum"

$$Z[G] = \int \mathcal{D}X^+ \mathcal{D}X \exp[-S(X, X^+)] \tag{14}$$

Here the differential $\mathcal{D}X^+\mathcal{D}X$ is just the product of bosonic and fermionic integration measures:

$$\mathcal{D}X^+\mathcal{D}X = \prod_{i=1}^{N} d\psi_i^* d\psi_i \prod_{j=1}^{M} \frac{d\phi_j^* d\phi_j}{2\pi i} \tag{15}$$

1) Argue that the action $S(X, X^+)$ is behaving like a commuting variable (i.e. it is a linear combination of usual complex numbers and products of even number of Grassmann variables).

2) Calculate $Z[G]$ if $N = 1, M = 1$. There is one fermionic degree of freedom ψ and one bosonic ϕ. The matrices G and X reduce to

$$G = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \; X = \begin{pmatrix} \psi \\ \phi \end{pmatrix} \tag{16}$$

The action $S(X, X^+)$ hence is

$$S(X, X^+) = \alpha\psi^*\psi + \delta\phi^*\phi + \psi^*\beta\phi + \phi^*\gamma\psi \tag{17}$$

where α, δ are usual commuting numbers; β, γ are Grassmann numbers.

3) Calculate $Z[G]$ for general N, M if $B = 0, \; C = 0$.

4) Again for a general N, M: make a linear transformation (shift in integration variables) and obtain an expression for the so-called *superdeterminant* $\det G$ of the matrix G:

$$\int \mathcal{D}X^{+}\mathcal{D}X \exp(-X^{+}GX) = \frac{1}{\det G} \tag{18}$$

$$\tag{19}$$

$\det G$ should be expressed through the usual determinants which are defined for commuting matrices. Check the limiting case of $B = 0, C = 0$.

Exercise III - Phonons (to be delivered before November 8th, 12:00)

Consider the following phonon Hamiltonian for a crystal in which there is one ion per unit cell at position $\mathbf{R}_i$

$$H_{ph} = \sum_i \frac{\mathbf{P}_i^2}{2M} + \frac{1}{2}\sum_{i\neq j} V(\mathbf{R}_i - \mathbf{R}_j). \tag{20}$$

Here M is the ionic mass and the ion coordinates are supposed to have small fluctuations around equilibrum lattice sites $\{\mathbf{R}_i^0\}$, $i = 1, \ldots, N$,

$$\mathbf{R}_i = \mathbf{R}_i^0 + \delta\mathbf{R}_i. \tag{21}$$

Within the harmonic approximation, the ion interaction V simplifies considerably. By introducing then the normal modes:

$$\delta\mathbf{R}_i = \frac{1}{\sqrt{NM}} \sum_{\mathbf{q},\lambda} Q_{\mathbf{q},\lambda} \boldsymbol{\varepsilon}_\lambda(\mathbf{q}) e^{i\mathbf{q}\mathbf{R}_i^0}, \tag{22}$$

$$\mathbf{P}_i = \sqrt{\frac{M}{N}} \sum_{\mathbf{q},\lambda} \Pi_{\mathbf{q},\lambda} \boldsymbol{\varepsilon}_\lambda(\mathbf{q}) e^{i\mathbf{q}\mathbf{R}_i^0}, \tag{23}$$

the Hamiltonian (20) can be written as

$$\frac{1}{2}\sum_{\mathbf{q},\lambda} \left(\Pi^\dagger_{\mathbf{q},\lambda}\Pi_{\mathbf{q},\lambda} + \omega^2_{\mathbf{q}\lambda} Q^\dagger_{\mathbf{q},\lambda} Q_{\mathbf{q},\lambda}\right), \tag{24}$$

where $\boldsymbol{\varepsilon}_\lambda(\mathbf{q}), \lambda = 1, 2, 3$ are polarization vectors satisfying $\boldsymbol{\varepsilon}^*_\lambda(\mathbf{q}) \cdot \boldsymbol{\varepsilon}_{\lambda'}(\mathbf{q}) = \delta_{\lambda,\lambda'}$, with $\boldsymbol{\varepsilon}^*_\lambda(\mathbf{q}) = \boldsymbol{\varepsilon}_\lambda(-\mathbf{q})$. The sums over $\mathbf{q}$ are restricted to the first Brillouin zone of the crystal and λ is the polarization.

1) Show that the usual commutation relations $[\delta\mathbf{R}_i^\mu, \mathbf{P}_j^\nu] = i\hbar\delta_{ij}\delta_{\mu\nu}$ imply

$$[Q_{\mathbf{q},\lambda}, \Pi_{\mathbf{q}',\lambda'}] = i\hbar\delta_{\mathbf{q},-\mathbf{q}'}\delta_{\lambda\lambda'}$$
$$[Q_{\mathbf{q},\lambda}, Q_{\mathbf{q}',\lambda'}] = [\Pi_{\mathbf{q},\lambda}, \Pi_{\mathbf{q}',\lambda'}] = 0$$

and interprete (24).

2) The creation and annihilation operators of phonons for the mode $\mathbf{q}$ and polarization λ can be defined by the usual relations

$$Q_{\mathbf{q},\lambda} = \left(\frac{\hbar}{2\omega_{\mathbf{q}\lambda}}\right)^{\frac{1}{2}} (b_{\mathbf{q}\lambda} + b^{\dagger}_{-\mathbf{q}\lambda}) \tag{25}$$

$$\Pi_{\mathbf{q},\lambda} = i\left(\frac{\hbar\omega_{\mathbf{q}\lambda}}{2}\right)^{\frac{1}{2}} (b^{\dagger}_{\mathbf{q}\lambda} - b_{-\mathbf{q}\lambda}). \tag{26}$$

with boson commutation relations

$$[b_{\mathbf{q}\lambda}, b^{\dagger}_{\mathbf{q}'\lambda'}] = \delta_{\mathbf{q}\mathbf{q}'}\delta_{\lambda\lambda'}$$
$$[b_{\mathbf{q}\lambda}, b_{\mathbf{q}'\lambda'}] = [b^{\dagger}_{\mathbf{q}\lambda}, b^{\dagger}_{\mathbf{q}'\lambda'}] = 0.$$

Show that the Hamiltonian (24) takes the form

$$\sum_{\mathbf{q},\lambda} \hbar\omega_{\mathbf{q}\lambda}\left(b^{\dagger}_{\mathbf{q}\lambda}b_{\mathbf{q}\lambda} + \frac{1}{2}\right) = \sum_{\mathbf{q}\lambda} \hbar\omega_{\mathbf{q}\lambda}\left(N_{\mathbf{q}\lambda} + \frac{1}{2}\right) \tag{27}$$

where $N_{\mathbf{q}\lambda} = b^{\dagger}_{\mathbf{q}\lambda}b_{\mathbf{q}\lambda}$ is the number operator for phonons. This is the phonon Hamiltonian in the harmonic approximation. What kind of terms would one get in (27) if one would take into account anharmonic terms when expanding the potential in eq. (20)?

3) Let us now introduce the operators

$$\varphi_\lambda\left(\mathbf{R}^0_i\right) = \frac{1}{\sqrt{V}}\sum_{\mathbf{q}}\left(b_{\mathbf{q}\lambda} + b^{\dagger}_{-\mathbf{q}\lambda}\right)e^{i\mathbf{q}\cdot\mathbf{R}^0_i}\,, \tag{28}$$

which become field operators $\varphi_\lambda(\mathbf{r})$ in continuum elasticity theory. The phonon Green's function is defined as

$$D_{\lambda\lambda'}\left(\mathbf{R}^0_i, t_1; \mathbf{R}^0_j, t_2\right) := -i\langle\Phi_0|T\left[\varphi_\lambda(\mathbf{R}^0_i, t_1)\,\varphi_{\lambda'}(\mathbf{R}^0_j, t_2)\right]|\Phi_0\rangle\,, \tag{29}$$

where $|\Phi_0\rangle$ is the ground state (the phonon vacuum for a harmonic crystal). Translational symmetry in space and time allows us to write

$$\begin{aligned} D_{\lambda\lambda'}\left(\mathbf{R}^0_i, t_1; \mathbf{R}^0_j, t_2\right) &= D_{\lambda\lambda'}\left(\mathbf{R}^0_i - \mathbf{R}^0_j, t_1 - t_2\right) \\ &= \int\frac{d\omega}{2\pi}\int_{BZ}\frac{d^3q}{(2\pi)^3}\,e^{i(\mathbf{q}\cdot\mathbf{r}-\omega t)}D_{\lambda\lambda'}(\mathbf{q},\omega)\,, \end{aligned} \tag{30}$$

where $\mathbf{r} = \mathbf{R}^0_i - \mathbf{R}^0_j$, $t = t_1 - t_2$ and the integration is restricted to the first Brillouin zone. $D_{\lambda\lambda'}(\mathbf{q},\omega)$ is the Fourier transform of

$$D_{\lambda\lambda'}(\mathbf{q}, t) = -i\langle\Phi_0|T\left[(b_{\mathbf{q}\lambda} + b^{\dagger}_{-\mathbf{q}\lambda})(t)(b_{-\mathbf{q}\lambda'} + b^{\dagger}_{\mathbf{q}\lambda'})\right]|\Phi_0\rangle, \tag{31}$$

where $b_{\mathbf{q}\lambda}(t) = e^{\frac{i}{\hbar}Ht}b_{\mathbf{q}\lambda}e^{-\frac{i}{\hbar}Ht}$.

Show that the one-phonon Green's function is given by

$$D_{\lambda\lambda'}(\mathbf{q},t) = \begin{cases} -i\delta_{\lambda\lambda'}e^{-i\omega_{\mathbf{q}\lambda}t} & , \quad t>0\,, \\ -i\delta_{\lambda\lambda'}e^{i\omega_{\mathbf{q}\lambda}t} & , \quad t<0, \end{cases} \tag{32}$$

where we have used the relations $|\Phi_0\rangle = |0\rangle$, $b_{\mathbf{q}\lambda}(t) = e^{-i\omega_{\mathbf{q}\lambda}t}b_{\mathbf{q}\lambda}$.

4) Show that the Fourier transform of $e^{-\eta|t|}D_{\lambda\lambda'}(\mathbf{q},t)$ is ($\eta>0$, infinitesimal)

$$D_{\lambda\lambda'}(\mathbf{q},\omega) = \delta_{\lambda\lambda'}\frac{2\omega_{\mathbf{q}\lambda}}{\omega^2-\omega_{\mathbf{q}\lambda}^2+i\eta'}\,, \tag{33}$$

where $\eta' = 2\omega_{\mathbf{q}\lambda}\eta > 0$. Consider now, *i.e.*, $t>0$. Determine the pole of $D_{\lambda\lambda'}(\mathbf{q},\omega)$ and discuss its physical interpretation.

5) If we would have considered electron-phonon interactions, we would have obtained the following expressions in the evaluation of the Green's function:

(a) $= \left(-\frac{i}{\hbar}\right)^2 \left[iG_{0\sigma}(\mathbf{k},\omega)\right]^2 \sum_\lambda \int \frac{d\mathbf{k}'}{(2\pi)^3} \int \frac{d\omega'}{2\pi} iG_{0\sigma'}(\mathbf{k}',\omega') iD_{0\lambda}(\mathbf{k}-\mathbf{k}',\omega-\omega') |g_{\mathbf{k}\mathbf{k}'\lambda}|^2$

(b) $= (-i)^2(-1)\left[iG_{0\sigma}(\mathbf{k},\omega)\right]^2 iD_{0\lambda}(0,0) g_{\mathbf{k}\mathbf{k}\lambda} \sum_{\sigma'} \int \frac{d\mathbf{k}'}{(2\pi)^3} \int \frac{d\omega'}{2\pi} iG_{0\sigma'}(\mathbf{k}',\omega') \overline{g_{\mathbf{k}'\mathbf{k}'\lambda}}$

(c) $= iD_{0\lambda}(\mathbf{k},\omega)$

(d) $= \left(-\frac{i}{\hbar}\right)^2 (-1)\left[iD_{0\lambda}(\mathbf{k},\omega)\right]^2 \sum_\sigma \int \frac{d\mathbf{k}'}{(2\pi)^3} \int \frac{d\omega'}{2\pi} iG_{0\sigma}(\mathbf{k}',\omega') iG_{0\sigma}(\mathbf{k}+\mathbf{k}',\omega+\omega') |g_{\mathbf{k}'\,\mathbf{k}+\mathbf{k}'\,\lambda}|^2$

Draw the corresponding Feynman diagrams and write the values of frequency and momentum in each line, using the definitions:
Associate with each electron thick line with $(\mathbf{k},\omega,\sigma)$ the propagator $iG_{0\sigma}(\mathbf{k},\omega)$.
Associate with each wavy phonon line with $(\mathbf{k},\omega,\lambda)$ the propagator $iD_{0\lambda}(\mathbf{k},\omega)$.
Discuss what happens in each vertex and ignore the prefactors.