Midterm Exam - "Statistical Field Theory"

November 6, 2007

Duration of the exam: 3 hours

1. Use a separate sheet for every exercise.

2. Write your name and initials in all sheets, on the first sheet also your student ID number.

3. Write clearly, unreadable work cannot be corrected.

4. You are NOT allowed to use any kind of books or lecture notes.

Exercise I - Quantum Ferromagnet: Magnons

The quantum Heisenberg ferromagnet is specified by the Hamiltonian

$$\hat{H} = -J \sum_{\langle mn \rangle} \hat{\mathbf{S}}_m \cdot \hat{\mathbf{S}}_n \tag{1}$$

where $J > 0$, $\hat{\mathbf{S}}_m$ represents the quantum mechanical spin operator at lattice site m, $\langle mn \rangle$ denotes summation over neighboring sites, and $\mathbf{S}_m^2 = S(S+1)$. Holstein and Primakoff have introduced a transformation in which the spin operators $\hat{S}^{\pm}$, $\hat{S}^z$ are specified in terms of bosonic creation and annihilation operators $a^\dagger$ and a:

$$\hat{S}_m^- = a_m^\dagger \left(2S - a_m^\dagger a_m\right)^{1/2}, \quad \hat{S}_m^+ = \left(2S - a_m^\dagger a_m\right)^{1/2} a_m, \quad \hat{S}_m^z = S - a_m^\dagger a_m.$$

Let us consider the problem in 1 dimension and put the lattice constant to unity. At low temperatures, for $S \gg 1/2$ we expect the deviations of the magnetization from its value at zero temperature to be very small, *i.e.* $S - \langle S_m^z \rangle = \langle a_m^\dagger a_m \rangle \ll S$. In this case, we may expand $\left(2S - a_m^\dagger a_m\right)^{1/2}$ in powers of $a_m^\dagger a_m$.

- (1.0) 1) Show that to first order in $a_m^\dagger a_m / S$ the Heisenberg Hamiltonian takes the form

  $$\hat{H} = -JNS^2 + JS \sum_m \left(a_{m+1}^\dagger - a_m^\dagger\right)\left(a_{m+1} - a_m\right) + higher\, order\, terms,$$

  where N is the total number of lattice sites.

- (1.0) 2) Keeping fluctuations at leading order in S, the quadratic Hamiltonian can be diagonalized by a Fourier transformation. In this case, it is convenient to impose periodic boundary conditions: $\hat{S}_{m+N} = \hat{S}_m$ and $a_{m+N} = a_m$. Perform the Fourier transformation and show that the Hamiltonian takes the form

  $$\hat{H} = -JNS^2 + \sum_k \hbar\omega_k a_k^\dagger a_k + higher\, order\, terms,$$

  where $\hbar\omega_k = 4JS\sin^2(k/2)$ represents the dispersion relation of the spin excitations. Calculate also the limit $k \to 0$ of the dispersion relation. These massless low-energy excitations, known as **magnons**, describe the elementary spin-wave excitations of the ferromagnet. Taking into account higher order terms, one finds the interactions between the magnons.

Exercise II - Mean-field shift and the spectral function

Consider the following partition sum Z, which describes interacting bosons with a source term I

$$Z[I] = \int d[\phi^*]d[\phi]e^{\sum_{i,j}\left\{\phi_i^* G_{i,j}^{-1}\phi_j - \frac{V_{i,j}}{2}\phi_i^*\phi_j^*\phi_j\phi_i + \phi_i^*\phi_i\delta_{i,j}I_j\right\}}$$

with the matrix $V_{i,j}$ describing the interaction. Notice that $V_{i,j}$ and $G_{i,j}^{-1}$ both can be inverted, that is

$$\begin{aligned}\sum_j V_{i,j}V_{j,k}^{-1} &= \delta_{i,k}\\ \sum_j G_{i,j}^{-1}G_{j,k} &= \delta_{i,k}.\end{aligned}$$

- (1.0) 1) Show that the partition sum can be written as

  $$Z[I] = \int d[\phi^*]d[\phi]d[\eta]e^{\sum_{i,j}\left\{\phi_i^* G_{i,j}^{-1}\phi_j + \frac{1}{2}\eta_i V_{i,j}\eta_j - \eta_i V_{i,j}\phi_j^*\phi_j + \eta_i\delta_{i,j}I_j + \frac{1}{2}I_i V_{i,j}^{-1}I_j\right\} + \frac{1}{2}\mathrm{Tr}\ln[-V_{i,j}]},$$

  where η is a new, real, bosonic field. The meaning of η will become clear in the next questions.

  Hint. Use the following identity for a path integral over the real field η

  $$e^{\frac{1}{2}\mathrm{Tr}\ln[-V_{i,j}]}\int d[\eta]e^{\frac{1}{2}\sum_{i,j}\left(\eta_i - \phi_i^*\phi_i + \sum_k I_k V_{k,i}^{-1}\right)V_{i,j}\left(\eta_j - \phi_j^*\phi_j + \sum_l V_{j,l}^{-1}I_l\right)} = 1.$$

  This is the famous Hubbard-Stratonovich transformation, about which we will learn more in the lectures to come.

- (1.0) 2) Show that the partition sum can be written in the following way

  $$Z[I] = e^{\frac{1}{2}\mathrm{Tr}\ln[-V_{i,j}]}\int d[\eta]e^{\sum_{i,j}\left\{\frac{1}{2}\eta_i V_{i,j}\eta_j + \eta_i\delta_{i,j}I_j + \frac{1}{2}I_i V_{i,j}^{-1}I_j\right\} - \mathrm{Tr}\ln\left[-G_{i,j}^{-1} + \sum_k \eta_k V_{k,j}\delta_{i,j}\right]}.$$

- (1.0) 3) Prove that $\langle\eta_i\rangle = \langle\phi_i^*\phi_i\rangle$.

  So far, we have shown that interactions cause a shift to the Green's function which is proportional to the interaction strength and the density of particles. This shift is called the mean-field shift. Consider now the following Green's function in frequency and momentum space

  $$G_{n,\mathbf{k}} = \frac{\hbar e^{i\omega_n\delta}}{i\hbar\omega_n - (\epsilon_{\mathbf{k}} - \mu + V\rho)},$$

  with $\epsilon_{\mathbf{k}}$ the kinetic energy, μ the chemical potential, V the interaction strength, ρ the total density of particles, and δ a small convergence factor. In the rest of the question we will be considering fermions, so $\omega_n = (2n+1)\pi/\hbar\beta$. Also note that both V and ρ are frequency and momentum independent.

- (1.0) 4) Show, by chosing the proper contours, that the Matsubara sum over the fermionic frequencies can be written as

$$\lim_{\delta\to 0}\frac{1}{\hbar\beta}\sum_n \frac{e^{i\delta\omega_n}}{i\omega_n-(\epsilon_{\mathbf{k}}-\mu+V\rho)/\hbar} = \lim_{\gamma\to 0}\frac{1}{2\pi i}\int_{-\infty}^{+\infty}dx\left\{\frac{1}{x+i\gamma-(\epsilon_{\mathbf{k}}-\mu+V\rho))/\hbar} - \frac{1}{x-i\gamma-(\epsilon_{\mathbf{k}}-\mu+V\rho)/\hbar}\right\}\frac{-1}{e^{\hbar\beta x}+1},$$

where we note that the limits for δ and γ come about for very different reasons.

- (1.0) 5) Finally, perform the integral on the right hand side and show that it gives

$$\lim_{\delta\to 0}\frac{1}{\hbar\beta}\sum_n \frac{e^{i\delta\omega_n}}{i\omega_n-(\epsilon_{\mathbf{k}}-\mu+Vn)/\hbar} = \frac{1}{e^{\beta(\epsilon_{\mathbf{k}}-\mu+Vn)}+1}$$

Hint.

$$\lim_{\gamma\to 0}\frac{\gamma}{x^2+\gamma^2}=\pi\delta(x)$$

Exercise III - One- and Two-Particle Green's functions

Consider an interacting fermionic system with partition function

$$\mathcal{Z}=\int\mathcal{D}\phi^*\mathcal{D}\phi\, e^{-\frac{S_0+S_{\rm I}}{\hbar}}$$

where S_0 is the free action and

$$S_{\rm I}=\frac{1}{2}\sum_{i,j}\int_0^{\hbar\beta}d\tau\int d\mathbf{x}_i d\mathbf{x}_j V(\mathbf{x}_i-\mathbf{x}_j)\,\phi_i^*(\mathbf{x}_i,\tau)\phi_j^*(\mathbf{x}_j,\tau)\phi_j(\mathbf{x}_j,\tau)\phi_i(\mathbf{x}_i,\tau)\,. \qquad (2)$$

The N-body Green's functions

$$\mathcal{G}(\mathbf{x}_1,\tau_1;\ldots;\mathbf{x}_n,\tau_n|\mathbf{x}'_1,\tau'_1;\ldots;\mathbf{x}'_n,\tau'_n) = -\langle T[\hat\psi_{\alpha_1}(\mathbf{x}_1,\tau_1)\hat\psi^\dagger_{\alpha'_1}(\mathbf{x}'_1,\tau'_1)\ldots\hat\psi_{\alpha_n}(\mathbf{x}_n,\tau_n)\hat\psi^\dagger_{\alpha'_n}(\mathbf{x}'_n,\tau'_n)]\rangle\,.$$

- (0.5) 1) What is the meaning of $T[\ldots]$ in the expression above?
- 2) Let us evaluate the one-body Green's function, i.e., the propagator,

$$\begin{aligned}\mathcal{G}(\mathbf{x},\tau|\mathbf{x}',\tau') &= -\langle T[\hat\psi_\alpha(\mathbf{x},\tau)\hat\psi^\dagger_{\alpha'}(\mathbf{x}',\tau')\rangle\\ &= -\langle\phi_\alpha(\mathbf{x},\tau)\phi^*_{\alpha'}(\mathbf{x}',\tau')\rangle\,.\end{aligned} \qquad (3)$$

  - (0.5) (a) Expand (3) perturbatively to first order in the interaction term, to yield $\mathcal{G}^{(1)}(\mathbf{x},\tau|\mathbf{x}',\tau')$.

- (1.0) (b) Using Wick's theorem, evaluate $\mathcal{G}^{(1)}(\mathbf{x}, \tau|\mathbf{x}', \tau')$ in terms of the non-interacting propagators $\mathcal{G}_0(\mathbf{x}, \tau|\mathbf{x}', \tau')$.
- (0.5) (c) Represent your result via Feynman diagrams.

- (0.5) 3) In (b) and (c), cancellations occurred in the expression for (3) by virtue of which some diagrams which might have contributed to the first-order Green's functions, did not. What kind of diagrams are those? Draw these cancelled contributions.