# Midterm Exam - "Statistical Field Theory"

November 4, 2008
Duration of the exam: 3 hours

*1. Use a separate sheet for every exercise.*
*2. Write your name and initials in all sheets, on the first sheet also your student ID number.*
*3. Write clearly, unreadable work cannot be corrected.*
*4. You are NOT allowed to use any kind of books or lecture notes.*

## Boson - fermion duality

The equivalence of the bosonic and the fermionic representation of the one-dimensional electron gas is exemplified by the computation of a correlation function.

### I: Fermionic representation:

To fix the notation please use

$$\psi(x) = \sqrt{\frac{1}{L}}\sum_k e^{ikx}\psi_k, \qquad \psi(x,\tau) = \sqrt{\frac{1}{L\hbar\beta}}\sum_{k,n} e^{ikx-i\omega_n\tau}\psi_{k,n},$$

where $L$ is the system size and $\beta = 1/k_BT$.

(0.5)**(1)** Write down the Hamiltonian of non-interacting spinless electrons moving in 1D in second quantized notion:
(a) In real space, in terms of the fermionic fields $\psi(x)$.
(b) In momentum representation, in terms of $\psi_k$.

(0.5)**(2)** Now, we would like to study a charge density excitation in the system at very low temperatures. Thus, we assume that the excitations occur only for electrons with energies close to $E_F$. This allows us to consider the spectrum to be linear around the Fermi points. Expand the spectrum around $k = \pm k_F$ up to linear order and show that after linearizing and introducing right-/left-moving fermionic operators $\psi_{+/-}$, the Hamiltonian reads

$$H_0 = \sum_{s=\pm 1}\sum_k s\hbar v_F k \psi^\dagger_{ks}\psi_{ks}. \qquad (1)$$

Notice that the momentum for the right (left) moving fermions is measured with respect to $k_F$ $(-k_F)$. The energy is measured with respect to $E_F$.

Further, we assume that the Hamiltonian is valid not only in the vicinity of the Fermi points but for all values of $k$. From now on we will put $v_F = 1$.

(1.0)**(3)** Show that in position representation the Hamiltonian (1) corresponds to the action ($\tau$ is the imaginary time)

$$\frac{1}{\hbar}S_0[\psi^\dagger,\psi] = \sum_{s=\pm 1}\int dx d\tau \psi^\dagger_s(x,\tau)(-is\partial_x + \partial_\tau)\psi_s(x,\tau). \qquad (2)$$

(1.0) **(4)** Employ the free fermion field integral

$$\mathcal{Z} = \int \mathcal{D}[\Psi^\dagger]\mathcal{D}[\Psi] \exp\left(-S_0[\Psi^\dagger, \Psi]/\hbar\right) \tag{3}$$

for a two-component field $\Psi = \binom{\psi_+}{\psi_-}$ comprising right- and left-moving fermions, to calculate the inverse of the Green's functions $G_\pm = -\langle \psi_\pm(x,\tau)\psi_\pm^\dagger(0,0)\rangle_{\psi_\pm}$ for the right- and left-moving fermions. Consider $\tau > 0$.

(1.0) **(5)** Prove that

$$G_\pm(x,\tau) = -\frac{1}{L\hbar\beta}\sum_{k,n}\frac{e^{ikx-i\omega_n\tau}}{-i\omega_n \pm k} \tag{4}$$

are Green's functions for the right- and left-moving fermions. Here $\omega_n$ are fermionic Matsubara frequencies.

(1.0) **(6)** Take the continuum limit $\frac{2\pi}{L}\sum_k \to \int dk$ to find for $x > 0$

$$G_\pm(x,\tau) = \mp\frac{i}{\hbar\beta}\sum_{n=-\infty}^{\infty}\Theta(\pm n)e^{\omega_n(\mp x - i\tau)}. \tag{5}$$

($\Theta(n) = 1$ for $n \geq 0$ and zero otherwise)

(1.0) **(7)** Show that in the zero-temperature limit this becomes

$$G_\pm(x,\tau) = \frac{1}{2\pi}\frac{1}{\pm ix - \tau} \tag{6}$$

(1.0) **(8)** Show that the fermionic correlation function

$$C_f(x,\tau) \equiv \langle \psi_-^\dagger(x,\tau)\psi_+(x,\tau)\psi_+^\dagger(0,0)\psi_-(0,0)\rangle_\Psi \tag{7}$$

can be written as the product of the Green's functions $G_\pm$ for the right- and left-moving fermions,

$$C_f(x,\tau) = G_+(x,\tau)G_-(x,\tau), \tag{8}$$

i.e.

$$C_f(x,\tau) = \frac{1}{(2\pi)^2}\frac{1}{x^2+\tau^2}. \tag{9}$$

## II- Bosonic representation

It is possible to describe the electronic problem in 1D in terms of bosons. Consider the "bosonic" electron density operator

$$\rho_s(q) = \sum_k \psi_{s,q+k}^\dagger\psi_{s,k}, \qquad [\rho_s(-q), \rho_{s'}(q')] = \delta_{s,s'}\delta_{q,q'}\frac{sqL}{2\pi}.$$

The Hamiltonian (1) can be rewritten in terms of these operators,

$$H_0 = \frac{2\pi}{L} \sum_{q>0,s} \rho_s(q)\rho_s(-q).$$

Defining $\theta(x) = \pi \int_{-\infty}^{x} dx' \rho(x')$ and using a combination of symmetry and dynamical arguments, one finds that the action in terms of the bosonic field $\theta(x,\tau)$ reads

$$S[\theta] = \frac{1}{2c} \int dx d\tau \left[ (\partial_\tau \theta)^2 + (\partial_x \theta)^2 \right] \tag{10}$$

where $c$ is a constant that we are going to determine later.

(0.5) **(9)** Express the field in its frequency/momentum Fourier representation to find

$$S[\theta] = \frac{1}{2c} \sum_{k,n} |\theta_{k,n}|^2 (k^2 + \omega_n^2). \tag{11}$$

(1.0) **(10)** Perform the Gaussian integral over $\theta$ to show that the correlation function

$$K(x,\tau) \equiv \langle \theta(x,\tau)\theta(0,0) - \theta(0,0)\theta(0,0) \rangle \tag{12}$$

can be written as

$$K(x,\tau) = \frac{c}{L\hbar\beta} \sum_{k,n} \frac{e^{ikx - i\omega_n \tau} - 1}{k^2 + \omega_n^2}. \tag{13}$$

Notice that now $\omega_n$ are bosonic Matsubara frequencies.

(1.0) **(11)** Use the continuum limit for the momenta and introduce a frequency cut-off $\omega_c = a^{-1}$ to find that in the zero-temperature limit

$$K(x,\tau) = \frac{c}{4\pi} \int_0^{a^{-1}} d\omega \frac{e^{-\omega(x - i\tau)} - 1}{\omega} + \text{c.c.} \tag{14}$$

For $x, \tau \gg a$, this integral can be performed (don't do it) and one finds that the correlation function behaves as

$$K(x,\tau) \simeq -\frac{c}{4\pi} \ln \left( \frac{x^2 + \tau^2}{a^2} \right). \tag{15}$$

(0.5) **(12)** Use that

$$\langle \exp A(x,\tau) \rangle_A = \exp \left( \frac{1}{2} \langle A(x,\tau)^2 \rangle_A \right) \tag{16}$$

if $\langle A^{2n+1} \rangle = 0$ and the results from (10) and (11) to compute the bosonic correlation function

$$C_b(x,\tau) \equiv \gamma^2 \langle \exp[2i\theta(x,\tau)] \exp[-2i\theta(0,0)] \rangle. \tag{17}$$

Choose the values of $c$ and $\gamma$ to obtain the equivalence between fermionic and bosonic representations, $C_f$ and $C_b$, of the correlation function.