Retake Exam - "Statistical Field Theory"

March 20th, 2007

Duration of the exam: 3 hours

1. *Use a separate sheet for every exercise.*
2. *Write your name and initials in all sheets, on the first sheet also your address and your student ID number.*
3. *Write clearly, unreadable work cannot be corrected.*
4. *You are NOT allowed to use any kind of books or lecture notes.*

Exercise 1: BCS-theory of the bosonic atom gas

During the lectures we studied BCS-theory for a fermionic gas of atoms. In this question, we will try to apply a similar transformation to a bosonic gas. Starting point is the spinless bosonic action for a homogeneous system

$$\begin{aligned} S[\phi^*,\phi] &= \int_0^{\hbar\beta} d\tau \int d\mathbf{x}\phi^*(\mathbf{x},\tau)\left[\hbar\partial_\tau - \frac{\hbar^2\nabla^2}{2m} - \mu\right]\phi(\mathbf{x},\tau) \\ &\quad +\frac{1}{2}\int_0^{\hbar\beta} d\tau \int d\mathbf{x}\; V_0\phi^*(\mathbf{x},\tau)\phi^*(\mathbf{x},\tau)\phi(\mathbf{x},\tau)\phi(\mathbf{x},\tau) \qquad (1) \end{aligned}$$

where we have to find a way to deal with the interaction term. An elegant way to do this, is by applying a Hubbard-Stratonovich transformation.

a) Perform a HS-transformation to the fields Δ and Δ^*, such that Δ is on average given by

$$\langle\Delta(\mathbf{x},\tau)\rangle = V_0\langle\phi(\mathbf{x},\tau)\phi(\mathbf{x},\tau)\rangle, \qquad (2)$$

and show that the resulting action in terms of the fields $\Delta^*, \Delta, \phi^*, \phi$ can be written in the form

$$\begin{aligned} S[\Delta^*,\Delta,\phi^*,\phi] &= -\int_0^{\hbar\beta} d\tau \int d\mathbf{x}\frac{|\Delta(\mathbf{x},\tau)|^2}{2V_0} \\ &\quad -\frac{\hbar}{2}\int_0^{\hbar\beta} d\tau d\tau' \int d\mathbf{x}d\mathbf{x}' \big(\phi^*(\mathbf{x},\tau),\phi(\mathbf{x},\tau)\big)\mathbf{G}^{-1}(\mathbf{x},\tau;\mathbf{x}',\tau')\begin{pmatrix}\phi(\mathbf{x}',\tau')\\ \phi^*(\mathbf{x}',\tau')\end{pmatrix}, \end{aligned}$$

with $\mathbf{G}^{-1}$ given by

$$\mathbf{G}^{-1} = \begin{pmatrix} G_0^{-1}(\mathbf{x},\tau;\mathbf{x}',\tau') & 0 \\ 0 & G_0^{-1}(\mathbf{x}',\tau';\mathbf{x},\tau)\end{pmatrix} - \frac{1}{\hbar}\begin{pmatrix} 0 & \Delta(\mathbf{x},\tau) \\ \Delta^*(\mathbf{x},\tau) & 0\end{pmatrix}\delta(\mathbf{x}-\mathbf{x}')\delta(\tau-\tau') \qquad (3)$$

In the case of a phase transition, the field Δ acquires a nonzero expectation value $\langle \Delta(\mathbf{x},\tau)\rangle = \Delta_0$. In the following, we will simply approximate the field $\Delta(\mathbf{x},\tau)$ by its average value Δ_0. This also means that we approximate the path integral over $\Delta(\mathbf{x},\tau)$ by its maximum contribution corresponding to Δ_0

b) Obtain the dispersion relation $\hbar\omega_{\mathbf{k}}$ from $\mathbf{G}^{-1}_{\mathbf{k},n}$, the Fourier transform of $\mathbf{G}^{-1}(\mathbf{x},\tau;\mathbf{x}',\tau')$. Hint: remember that in Fourier space:

$$-\hbar G^{-1}_{0,\mathbf{k},n} = \mp i\hbar\omega_n + \epsilon_{\mathbf{k}} - \mu, \tag{4}$$

where the correct sign in front of ω_n depends on the particular time ordering and perform the analytic continuation $i\omega \to \omega$.

Since the obtained action is quadratic in the bosonic fields ϕ^*, ϕ, we can perform the path integral over these fields exactly, so that we obtain the partition sum Z.

c) Show that

$$Z = \mathrm{Exp}\left\{\beta V \frac{|\Delta_0|^2}{2V_0} - \frac{1}{2}\mathrm{Tr}\log[-\mathbf{G}^{-1}]\right\}, \tag{5}$$

where V is the volume of our system.

The information about a second order phase transition is hidden in the part of $\Omega = -\log[Z]/\beta$ that is quadratic in the order parameter $|\Delta_0|$, since that part can tell us, whether the minimum $|\Delta_0| = 0$ is a stable one. For this reason we will treat the partition sum in the following way. First we split up $\mathbf{G}^{-1}$ in a diagonal and an off-diagonal part

$$\mathbf{G}^{-1} = \begin{pmatrix} G_{0,11} & 0 \\ 0 & G_{0,22} \end{pmatrix} - \begin{pmatrix} 0 & \Sigma_{12} \\ \Sigma_{21} & 0 \end{pmatrix} \equiv \mathbf{G}_0^{-1} - \mathbf{\Sigma} \tag{6}$$

and thus

$$\log[-\mathbf{G}^{-1}] = \log[-\mathbf{G_0}^{-1}(1-\mathbf{G_0}\mathbf{\Sigma})] = \log[-\mathbf{G_0}^{-1}] + \log[1-\mathbf{G_0}\mathbf{\Sigma}]. \tag{7}$$

Since we are interested of the part which is quadratic in Δ_0 we expand the logarithm up to second order in $\mathbf{\Sigma}$ and keep only this quadratic term in $\mathbf{\Sigma}$. This leads to the following contribution to Z

$$\frac{1}{4}\mathrm{Tr}[\mathbf{G_0}\mathbf{\Sigma}\mathbf{G_0}\mathbf{\Sigma}] = \frac{1}{2}\mathrm{Tr}[G_{0,11}\Sigma_{12}G_{0,22}\Sigma_{21}] = \frac{|\Delta_0|^2}{2\hbar^2}\sum_{\mathbf{k},n} \frac{-\hbar}{-i\hbar\omega_n + \epsilon_{\mathbf{k}} - \mu}\,\frac{-\hbar}{i\hbar\omega_n + \epsilon_{\mathbf{k}} - \mu} \tag{8}$$

d) Perform the Matsubara summation on the right-hand-side of eq. (8) and show that the system undergoes a second-order phase transition at the condition:

$$\frac{1}{V_0} + \frac{1}{V}\sum_{\mathbf{k}} \frac{1+2N(\epsilon_{\mathbf{k}})}{2(\epsilon_{\mathbf{k}} - \mu)} = 0 \tag{9}$$

Hint:

$$\lim_{\eta\downarrow 0}\frac{1}{\hbar\beta}\sum_n \frac{-\hbar e^{i\omega_n\eta}}{-i\hbar\omega_n+\epsilon_{\mathbf{k}}-\mu} = -\frac{1}{e^{\beta(\epsilon_{\mathbf{k}}-\mu)}-1}\equiv -N(\epsilon_{\mathbf{k}}) \tag{10}$$

$$\lim_{\eta\downarrow 0}\frac{1}{\hbar\beta}\sum_n \frac{-\hbar e^{i\omega_n\eta}}{i\hbar\omega_n+\epsilon_{\mathbf{k}}-\mu} = -\frac{1}{e^{\beta(\epsilon_{\mathbf{k}}-\mu)}-1}-1\equiv -N(\epsilon_{\mathbf{k}})-1 \tag{11}$$

$$\tag{12}$$

In general the behavior of Ω as a function of $|\Delta_0|$ is given by

$$\Omega = c_0 + c_2|\Delta_0|^2 + c_4|\Delta_0|^4 + ... \tag{13}$$

In the previous question we calculated microscopically the prefactor c_2 in order to get information about the second order phase transition in the system.

e)To which order should we calculate $\Omega(|\Delta_0|)$ in order to be able to describe a first order phase transition? Include in your answer the signs (positive or negative) of the various prefactors and include also sketches of $\Omega(|\Delta_0|)$.

Exercise 1: Charge-Density-Wave Instability

The Peierls instability is a phenomenon in one-dimensional crystals comprised of ions and electrons. It manifests itself in the appearance of an *electron charge-density-wave* (CDW), i.e. periodically modulated electron charge density in space. In such a way, apart from the underlying periodical lattice of ions, a new periodical structure appears, with a period different from the period of the ion lattice.

Let us consider an action for a one dimensional gas of spinless electrons

$$S_{el}[\psi,\bar{\psi}] = \int_0^\beta d\tau \int_0^L dx\bar{\psi}\left(\partial_\tau - \frac{1}{2m}\partial_x^2 - \mu\right)\psi \tag{14}$$

where L is the linear size of the system, $\beta = 1/T$, with T being the temperature, in the system of units where $k_B = 1, \hbar = 1$. Electrons also interact with ions in the one-dimensional lattice. The action for the system of ions is given by

$$S_{ph}[u] = \frac{\rho}{2}\int_0^\beta d\tau \int_0^L dx\left\{(\partial_\tau u)^2 + c^2(\partial_x u)^2\right\} \tag{15}$$

where $u(x,\tau)$ denotes the static bosonic displacement field (describing *phonons*) and $\rho > 0$ is the density of ions and c is the sound velocity. The coupling of electrons to the lattice vibrations of the lattice of ions is described by

$$S_{el-ph}[\psi,\bar{\psi},u] = g\int_0^\beta d\tau \int_0^L dx\bar{\psi}\psi\partial_x u \tag{16}$$

And the full interacting theory is described by the action

$$S = S_{el} + S_{ph} + S_{el-ph} \tag{17}$$

We cannot solve it exactly and will rely instead on perturbation theory.

a) As a first step, integrate out the fermionic degrees of freedom ψ and thereby obtain an effective action $S_{eff}[u]$ for the displacement field $u(x,\tau)$. Assuming that the electron-phonon coupling constant g is small, expand the action up to second order in u. You will find that the coefficient by $g^2|u(q,i\Omega_n)|^2$ in momentum space involves the *density-density response function*

$$\chi(q,i\Omega_n) = -\frac{1}{2\beta L}\sum_{km}\left[G_0(k+q,i\omega_m+i\Omega_n)G_0(k,i\omega_m)+G_0(k-q,i\omega_m-i\Omega_n)G_0(k,i\omega_m)\right]$$

with $\Omega_n = 2\pi n/\beta$ and $\omega_n = (2n+1)\pi/\beta$ being respectively bosonic and fermionic Matsubara frequencies.

b) Now you have to find a saddle-point of the effective action $S_{eff}[u]$. One may look for a homogeneous displacement field, i.e. $u(x,\tau) \equiv u_0$. However, here it is not necessarily the best solution. Show that the static solution $u_0(x,\tau) \equiv u_0\cos(2k_F x+\varphi)$ is energetically favorable (i.e. $S[u = u_0\cos(2k_F x+\varphi)] < S[u=0]$) below a certain critical temperature T_c. Thus, at low temperatures, the system is unstable towards the formation of a static sinusoidal lattice distortion. Calculate the critical temperature T_c, by using the following approximation for the response function $\chi(2k_F,0) \approx \ln(\beta\omega_D)/(4\pi v_F)$, where ω_D is the Debye frequency and $v_F = \pi n_e/m_e$ is the Fermi velocity, $k_F = m_e v_F$ is the Fermi momentum, n_e the density of electrons.

c) If you have already obtained an answer for T_c as a function of g, can you in principle obtain it directly by means of a perturbation theory in g? If not, why? What is the connection between the result you obtained and the BCS theory of superconductivity? What is the period of the lattice?