INSTITUTE FOR THEORETICAL PHYSICS, FACULTY OF PHYSICS AND ASTRONOMY, UU.
MADE AVAILABLE IN ELECTRONIC FORM BY THE $\mathcal{TBC}$ OF A–ESKWADRAAT
IN 2004/2005, THE COURSE NS-TP402M WAS GIVEN BY PROF. DR. CRISTIANE DE MORAIS SMITH.

Statistical Field Theory (retake exam) (NS-TP402M)
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- Duration of the exam: 3 hours.
- You are NOT allowed to use any kind of books or lecture notes.
- Every exercise is worth 1.0 points, there are 10 exercises in total.

Question 1. Electron-phonon coupling *(3 points)*

Consider the phonon Hamiltonian

$$\hat{H}_{\mathrm{ph}} = \sum_{\mathbf{q},j} \hbar\omega_q \hat{a}^\dagger_{\mathbf{q},j}\hat{a}_{\mathbf{q},j} + const., \tag{1}$$

where ω_q is the phonon dispersion (here we assumed to depend only on the modulus of the momentum, $|\mathbf{q}| = q$) and the index $j = 1, 2, 3$ accounts for the fact that the lattice ions can oscillate in three dimensions in space (i.e. there are three linearly independent oscillator modes). The electron-phonon Hamiltonian reads

$$\hat{H}_{\mathrm{el-ph}} = \hbar\gamma \sum_{\mathbf{q},j} \frac{iq_j}{(2M\omega_q)^{1/2}} \hat{n}_{\mathbf{q}}(\hat{a}^\dagger_{-\mathbf{q},j} + \hat{a}_{\mathbf{q},j}). \tag{2}$$

Here, $\hat{n}_{\mathbf{q}} = \sum_{\mathbf{k}} \hat{c}^\dagger_{\mathbf{k}+\mathbf{q}}\hat{c}_{\mathbf{k}}$ denotes the electronic density expressed in terms of fermion creation and annihilation operators, and the electron spin has been neglected for simplicity.

a) Formulate the coherent state action of the electron-phonon system by introducing a Grassmann field ψ (a complex field ϕ) to represent electron (phonon) operators, and obtain the coherent state field integral

$$\mathcal{Z} = \int D[\bar{\psi},\psi] \int D[\bar{\phi},\phi] e^{-(S_{\mathrm{el}}[\bar{\psi},\psi] + S_{\mathrm{ph}}[\bar{\phi},\phi] + S_{\mathrm{el-ph}}[\bar{\psi},\psi,\bar{\phi},\phi])/\hbar}, \tag{3}$$

where we leave $S_{\mathrm{el}}[\bar{\psi},\psi]$ unspecified. Write down $S_{\mathrm{ph}}[\bar{\phi},\phi]$ and $S_{\mathrm{el-ph}}[\bar{\psi},\psi,\bar{\phi},\phi]$ explicitly using Matsubara frequencies.

b) Integrate out the phonon fields to obtain

$$S_{\mathrm{eff}}[\bar{\psi},\psi] = S_{\mathrm{el}}[\bar{\psi},\psi] - \frac{\hbar\gamma^2}{2M} \sum_{n,\mathbf{q}} \frac{q^2}{\omega_q} \frac{1}{i\omega_n + \omega_q} \rho_{\mathbf{q},n}\rho_{-\mathbf{q},-n}, \tag{4}$$

where $\rho_{\mathbf{q},n} = (1/\sqrt{\hbar\beta}) \sum_{\mathbf{k},m} \bar{\psi}_{\mathbf{k}+\mathbf{q},m}\psi_{\mathbf{k},m+n}$.

c) Show that this is equivalent to

$$S_{\mathrm{eff}}[\bar{\psi},\psi] = S_{\mathrm{el}}[\bar{\psi},\psi] - \frac{\hbar\gamma^2}{2M} \sum_{n,\mathbf{q}} \frac{q^2}{\omega_n^2 + \omega_q^2} \rho_{\mathbf{q},n}\rho_{-\mathbf{q},-n}. \tag{5}$$

Replacing ω_n by $-i\omega$, give the condition for ω with respect to ω_q for getting an attractive interaction. Do you know which physical system has an attractive electron-electron interaction intermediated by the phonons? Which are the physical consequences of this interaction?

Question 2. BEC of polaritons *(7 points)*

Polaritons are linear superpositions of excitons and photons (remember that an exciton is an electron-hole bound state). The Hamiltonian of polaritons is given by

$$\hat{H}_{tot} = \hat{H}_{exc} + \hat{H}_{ph} + \hat{H}_{exc-ph}, \tag{6}$$

where $\hat{H}_{exc}$ is an excitonic Hamiltonian, $\hat{H}_{ph}$ is a photonic Hamiltonian, $\hat{H}_{exc-ph}$ is a Hamiltonian of exciton-photon interaction. The Hamiltonian of $2D$ excitons in the infinite homogeneous system is given by

$$\hat{H}_{exc} = \sum_{\mathbf{P}} \varepsilon_{ex}(P)\hat{b}^{\dagger}_{\mathbf{P}}\hat{b}_{\mathbf{P}} + \frac{1}{2A}\sum_{\mathbf{P},\mathbf{P}',\mathbf{q}} U_{\mathbf{q}}\hat{b}^{\dagger}_{\mathbf{P}+\mathbf{q}}\hat{b}^{\dagger}_{\mathbf{P}'-\mathbf{q}}\hat{b}_{\mathbf{P}}\hat{b}_{\mathbf{P}'}, \tag{7}$$

where $\hat{b}^{\dagger}_{\mathbf{P}}$ and $\hat{b}_{\mathbf{P}}$ are excitonic creation and annihilation operators obeying bosonic commutation relations. In the first term, $\varepsilon_{ex}(P)$ is the energy dispersion of a single exciton in a quantum well. In the interaction term, A is the macroscopic quantization area and $U_{\mathbf{q}}$ is the Fourier transform of the exciton-exciton pair repulsion potential. For small wave vectors the pair exciton-exciton repulsion can be approximated as a contact potential $U_{\mathbf{q}} \simeq U_0 \equiv U$. The Hamiltonian of non-interacting photons in a semiconductor microcavity is given by:

$$\hat{H}_{ph} = \sum_{\mathbf{P}} \varepsilon_{ph}(P)\hat{a}^{\dagger}_{\mathbf{P}}\hat{a}_{\mathbf{P}}, \tag{8}$$

where $\hat{a}^{\dagger}_{\mathbf{P}}$ and $\hat{a}_{\mathbf{P}}$ are photonic creation and annihilation Bose operators, and $\varepsilon_{ph}(P)$ is the cavity photon spectrum. The Hamiltonian of a harmonic exciton-photon coupling has the form:

$$\hat{H}_{exc-ph} = \hbar\sum_{\mathbf{P}}(\Omega_R\hat{a}^{\dagger}_{\mathbf{P}}\hat{b}_{\mathbf{P}} + h.c.), \tag{9}$$

where the exciton-photon coupling energy represented by the Rabi constant $\hbar\Omega_R$ depends on the overlap between the exciton and photon wavefunction.

a) Perform a diagonalization of the total Hamiltonian $\hat{H}_{tot}$ given in Eq. (6), neglecting the interaction term in Eq. (7) and show that the diagonalized Hamiltonian has the form:

$$\hat{H}_0 = \sum_{\mathbf{P}} \varepsilon_{LP}(P)\hat{p}^{\dagger}_{\mathbf{P}}\hat{p}_{\mathbf{P}} + \sum_{\mathbf{P}} \varepsilon_{UP}(P)\hat{u}^{\dagger}_{\mathbf{P}}\hat{u}_{\mathbf{P}}, \tag{10}$$

where $\hat{p}^{\dagger}_{\mathbf{P}}$ and $\hat{u}^{\dagger}_{\mathbf{P}}$ are bosonic creation operators for the lower and upper polaritons, respectively. The energy spectra of the upper/lower polaritons are

$$\varepsilon_{UP/LP}(P) = \frac{\varepsilon_{ph}(P) + \varepsilon_{ex}(P)}{2} \pm \frac{1}{2}\sqrt{(\varepsilon_{ph}(P) - \varepsilon_{ex}(P))^2 + 4|\hbar\Omega_R|^2}, \tag{11}$$

which implies a splitting between the upper and lower states at $P = 0$ of $2\hbar\Omega_R$, known as the Rabi splitting. Substituting the polaritonic representation of the excitonic and photonic operators into the total Hamiltonian $\hat{H}_{tot}$, the Hamiltonian of lower polaritons is obtained (don't do it):

$$\hat{H}_{tot} = \sum_{\mathbf{P}} \varepsilon_{LP}(P)\hat{p}^{\dagger}_{\mathbf{P}}\hat{p}_{\mathbf{P}} + \frac{1}{2A}\sum_{\mathbf{P},\mathbf{P}',\mathbf{q}} U_{\mathbf{P},\mathbf{P}',\mathbf{q}}\hat{p}^{\dagger}_{\mathbf{P}+\mathbf{q}}\hat{p}^{\dagger}_{\mathbf{P}'-\mathbf{q}}\hat{p}_{\mathbf{P}}\hat{p}_{\mathbf{P}'}. \tag{12}$$

Assuming $U_{\mathbf{P},\mathbf{P}',\mathbf{q}} = U_{\text{eff}}$, the effective Hamiltonian becomes

$$\hat{H}_{\text{eff}} = \sum_{\mathbf{P}} \frac{\hbar^2P^2}{2M_{\text{eff}}}\hat{p}^{\dagger}_{\mathbf{P}}\hat{p}_{\mathbf{P}} + \frac{U_{\text{eff}}}{2A}\sum_{\mathbf{P},\mathbf{P}',\mathbf{q}} \hat{p}^{\dagger}_{\mathbf{P}+\mathbf{q}}\hat{p}^{\dagger}_{\mathbf{P}'-\mathbf{q}}\hat{p}_{\mathbf{P}}\hat{p}_{\mathbf{P}'}, \tag{13}$$

where M_{eff} is the effective mass of a polariton.

b) Transform the Hamiltonian Eq. (13) to the real space representation and show that

$$\hat{H} = \int d\mathbf{r}\hat{\phi}^\dagger(\mathbf{r})\left(-\frac{\hbar^2\nabla^2}{2M_{\text{eff}}}\right)\hat{\phi}(\mathbf{r}) + \frac{U_{\text{eff}}}{2}\int d\mathbf{r}\hat{\phi}^\dagger(\mathbf{r})\hat{\phi}^\dagger(\mathbf{r})\hat{\phi}(\mathbf{r})\hat{\phi}(\mathbf{r}), \tag{14}$$

where $\hat{\phi}^\dagger(\mathbf{r})$ and $\hat{\phi}(\mathbf{r})$ are real space bosonic field operators of creation and annihilation of polaritons, correspondingly.

Taking into account the stress induced harmonic trap by adding to the Hamiltonian $V_{\text{eff}}(r) = \frac{1}{2}\gamma r^2$, with certain parameter γ, the effective Hamiltonian for trapped polaritons will look exactly like the Hamiltonian of a weakly-interacting dilute 2D Bose gas in a confining trap:

$$\hat{H} = \int d\mathbf{r}\hat{\phi}^\dagger(\mathbf{r})\left(-\frac{\hbar^2\nabla^2}{2M_{\text{eff}}} + V_{\text{eff}}(r)\right)\hat{\phi}(\mathbf{r}) + \frac{U_{\text{eff}}}{2}\int d\mathbf{r}\hat{\phi}^\dagger(\mathbf{r})\hat{\phi}^\dagger(\mathbf{r})\hat{\phi}(\mathbf{r})\hat{\phi}(\mathbf{r}). \tag{15}$$

Although Bose-Einstein condensation (BEC) cannot happen in a 2D homogeneous ideal gas at non-zero temperature, in a harmonic trap it can occur in two dimensions below a critical temperature $T_c^0 > 0$.

c) Write down the partition function and the action for the grand-canonical ensemble with the Hamiltonian Eq. (15) and chemical potential μ. Now, assume that the polaritons are undergoing a Bose-Einstein condensation. Using the Bogoliubov approximation, you can write

$$\phi(\mathbf{r},\tau) = \phi_0(\mathbf{r}) + \varphi(\mathbf{r},\tau), \tag{16}$$

where $\varphi(\mathbf{r},\tau)$ denotes the fluctuations. Derive the Gross-Pitaevskii equation for the polariton condensate and solve it in the Thomas-Fermi approximation (neglect the kinetic energy term). What is the size of the condensate? How can you find the number of particles in the condensate (you do not have to evaluate the expressions)?

d) Derive the quadratic part of the action in the fluctuations and show that you obtain

$$S_{\varphi^2} = -\frac{\hbar}{2}\int_0^{\hbar\beta} d\tau d\tau' \int d\mathbf{r}d\mathbf{r}' \Big[\varphi^*(\mathbf{r},\tau), \varphi(\mathbf{r},\tau)\Big]\mathbf{G}^{-1}(\mathbf{r},\tau;\mathbf{r}',\tau')\begin{bmatrix}\varphi(\mathbf{r}',\tau')\\ \varphi^*(\mathbf{r}',\tau')\end{bmatrix},$$

and

$$\mathbf{G}^{-1}(\mathbf{r},\tau;\mathbf{r}',\tau') = \begin{pmatrix}\mathbf{G}_0^{-1}(\mathbf{r},\tau;\mathbf{r}',\tau') & 0\\ 0 & \mathbf{G}_0^{-1}(\mathbf{r}',\tau';\mathbf{r},\tau)\end{pmatrix}$$
$$-\frac{1}{\hbar}\begin{pmatrix}2U_{\text{eff}}|\phi_0|^2 & U_{\text{eff}}\phi_0^2\\ U_{\text{eff}}\phi_0^{*2} & 2U_{\text{eff}}|\phi_0|^2\end{pmatrix}\delta(r-r')\delta(\tau-\tau')$$

with $\mathbf{G}_0^{-1}(\mathbf{r},\tau;\mathbf{r}',\tau')$ given by

$$\mathbf{G}_0^{-1}(\mathbf{x},\tau;\mathbf{x}',\tau') = -\frac{1}{\hbar}\left(\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2\nabla^2}{2M_{\text{eff}}} + V_{\text{eff}}(r) - \mu\right)\delta(r-r')\delta(\tau-\tau'). \tag{17}$$

e) Consider the condensate in the middle of the trap and assume the system to be locally homogeneous. By performing a Fourier transformation of $\mathbf{G}^{-1}$ using plane waves (homogeneous system) and Matsubara frequencies you would obtain (don't do it)

$$-\hbar\mathbf{G}^{-1}(\mathbf{k},i\omega_n) = \begin{pmatrix}-i\hbar\omega_n + \epsilon_{\mathbf{k}} + U_{\text{eff}}|\phi_0|^2 & U_{\text{eff}}\phi_0^2\\ U_{\text{eff}}\phi_0^{*2} & i\hbar\omega_n + \epsilon_{\mathbf{k}} + U_{\text{eff}}|\phi_0|^2\end{pmatrix}.$$

Why do you have opposite signs in front of $i\hbar\omega_n$? Write the expression for $\epsilon_{\mathbf{k}}$. Why do you have here $U_{\text{eff}}|\phi_0|^2$ without the factor 2 in the diagonal terms?

f) Find the spectrum of physical modes in the system by making an analytic continuation (or if you like Wick rotation) $i\omega_n \to \omega_{\mathbf{k}}$. This brings us back to the Feynman path integral formulation in real time, in which $\hbar\omega_{\mathbf{k}}$ stands for the energy spectrum or physical modes of the system. You must show that

$$\hbar\omega_{\mathbf{k}} = \pm\sqrt{\epsilon_{\mathbf{k}}^2 + 2U_{\text{eff}}|\phi_0|^2\epsilon_{\mathbf{k}}} \tag{18}$$

g) What happens with the energy spectrum in the limit $\mathbf{k} \to 0$? Can you get any conclusions involving broken symmetries out of that?