General Relativity 2011/12 – Midterm Exam, 11 Nov 2011, 14:00-16:00h

You must obtain at least 10 of the 20 points to pass the exam. The use of auxiliary materials such as books, notes, calculators, laptops etc. is not permitted. Please hand in all sheets you used for calculations.

Problem 1 - Curvature tensors (6 points)

(a) Derive the number of algebraically independent components of the Riemann tensor, the Ricci tensor and the Ricci scalar as a function of the spacetime dimension d, taking into account their symmetry properties. (*1.5 points*)

(b) In spacetimes of fewer than 4 dimensions, there exist simple formulas for the Riemann tensor. Using the results derived in (a), argue that in *two* dimensions the usual relation expressing the Ricci tensor $R_{\rho\sigma}$ as a linear function of the components $R_{\kappa\lambda\mu\nu}$ of the Riemann tensor can be inverted to express the Riemann tensor as a function of the $R_{\rho\sigma}$. Derive this tensorial expression explicitly, and show that it depends on the $R_{\rho\sigma}$ only through the Ricci scalar R. (*2 points*)

(c) Using the results derived in (a), argue that in *three* dimensions the usual relation expressing the Ricci tensor $R_{\rho\sigma}$ as a linear function of the components $R_{\kappa\lambda\mu\nu}$ of the Riemann tensor can be inverted to express the Riemann tensor as a function of the $R_{\rho\sigma}$. Derive this tensorial expression explicitly. (Hint: find an algebraic ansatz for $R_{\kappa\lambda\mu\nu}$ which has the correct tensorial and symmetry properties.) (*2.5 points*)

Problem 2 - Covariant derivative and geodesics (8.5 points)

(a) Derive explicitly the transformation law under a coordinate transformation $x^\mu \mapsto x'^\mu(x^\nu)$ of the connection coefficients $\Gamma^\lambda_{\mu\nu}$ associated with an arbitrary covariant derivative ∇. (*2.5 points*)

(b) For the space with metric

$$ds^2 = dr^2 + r^2 d\theta^2, \quad r \geq 0,\ 0 \leq \theta \leq 2\pi, \tag{1}$$

compute the coefficients of the metric-compatible connection (the Christoffel symbols) and thus determine the two equations that follow from the geodesic equation. (*2 points*)

(c) With the help of (b), derive two first integrals of the geodesic equations, namely,

$$r^2 \frac{d\theta}{ds} = C, \quad \left(\frac{dr}{ds}\right)^2 + r^2\left(\frac{d\theta}{ds}\right)^2 = 1, \tag{2}$$

where C is a constant. Use these results to derive a *first-order* differential equation for $r(\theta)$. (In other words, eliminate s as a parameter and replace it by θ.) (*2 points*)

(d) Using the fact that (1) just describes flat two-dimensional Euclidean space, write down the general equation for a geodesic in r, θ coordinates, and verify that it satisfies the first-order equation for $r(\theta)$ you derived in (c). (*2 points*)

Problem 3 - Induced metric (5.5 points)

In three-dimensional Minkowski space with metric $ds^2 = -dt^2 + dx^2 + dy^2$, consider the two-dimensional surface S defined by the equation

$$t^2 - x^2 - y^2 = c^2, \tag{3}$$

where c is a positive constant.

(a) Compute the induced metric on S as a function of x and y. What is the signature of the induced metric (i.e. the number of positive and negative eigenvalues), and is it the same everywhere on S? (*2.5 points*)

(b) Find a set of coordinates which make the metric on S diagonal. Do not attempt to do this by brute force, but consider the geometry of the surface inside Minkowski space and make an ansatz in terms of suitable angles. Compute the diagonal metric explicitly. (*3 points*)