NS-TP428M

GR-exam February the fifth, 2010, 9-12 am

The use of auxilary material such as notes, books, calculators, laptops, etc. is not allowed. In the problems below units are used such that the constant c in Einsteins special relativity is equal 1. Each of the 13 (sub)-questions carry the same weight.

(1) Let $T^{\mu\nu}$ denote the energy-momentum tensor. How does it transform under a general coordinate transformation $x^\mu \to y^\mu(x^\nu)$

(2) Write down the energy momentum tensor for a perfect fluid, characterized by its pressure and density and the four-velocity U^μ.

(3) Consider a satellite in a circular orbit around the Earth, at height H above the surface of the Earth. Assume that the equations of motion for the satellite are well approximated by the Newtonian equations of motion and that the shape of the Earth is spherical. Write down an expression for $d\tau_s/d\tau_e$, where τ_s and τ_e are the proper times for clocks at rest in the satellite and at the surface of the Earth, respectively. The expression should contain the acceleration g at the surface of the Earth, the radius R of the Earth and H.

(4) Consider an inertial system with coordinates $X^\mu = (T, X, Y, Z)$. Now make a coordinate transformation to an accelerated coordinate system $x^\nu = (t, x, y, z)$ defined by:

$$X = x + \frac{1}{2}gt^2, \quad Y = y, \quad Z = z, \quad T = t.$$

(4a) Find the expression for $d\tau^2$ expressed in the coordinates x^μ for $t < 1/g$.

(4b) Recall that the spatial distance $d\sigma$ at time t between two observers with coordinates (x, y, z, t) and $(x + dx, y + dy, z + dz, t)$ is defined as the spatial distance measured by standard measuring rods in a "local elevator", i.e. in an inertial system at rest at time t relative to the observer with coordinates (x, y, z, t). Find an expression for $d\sigma^2$.

(4c) Write down the equations for a freely falling massive particle in the coordinate system x^μ, and solve the equations with initial conditions $x(t) = y(t) = z(t) = 0$ at $t = 0$, and $\dot{x}(t) = v < 1$, $\dot{y}(t) = \dot{z}(t) = 0$ at $t = 0$ (a "dot" above a variable means differentiation signifies the derivative with respect to the coordinate time t).

(4d) Write down and solve in the coordinate system x^μ the equations of motion for a light ray passing through the origin of coordinate system at $t = 0$ in the direction of the positive x-axis. How do you check in a trivial way that your results in (c) and (d) are correct?

(5) Write down the static metric corresponding to the Schwarzschild solution with a point mass M in standard spherical coordinates $x^\mu = (t, r, \theta, \phi)$, the mass positioned at $r = 0$.

(6) Consider a massive particle moving freely in a Schwarzschild geometry, using the static metric.

(6a) Write down the (four) equations of motion for the massive particle.

(6b) Show that they are consistent with

$$\theta = \pi/2, \quad r^2\dot{\phi} = H, \quad \Big(1 - \frac{2MG}{r}\Big)\dot{t} = L,$$

where H and L are integration constants and the dot denotes differentiation with respect to proper time.

(6c) Use the line element from (5) to derive

$$\dot{r}^2 + V(r) = L^2, \quad V(r) = \Big(1 - \frac{2MG}{r}\Big)\Big(1 + \frac{H^2}{r^2}\Big)$$

for a motion characterized by $\theta = \pi/2$ and H and L.

(6d) Consider now a circular motion. This leads to the requirement that $\dot{r} = 0$ and $\ddot{r} = 0$. Show (e.g. from (6c), differentiating with respect to proper time) that r is then equal

$$r^{\pm}_{\text{circular}} = \frac{MG}{2}\Big(\frac{H}{MG}\Big)^2\Big(1 \pm \sqrt{1 - 12\Big(\frac{MG}{H}\Big)^2}\Big).$$

For a given mass M the radius r_{circular} cannot be arbitrarily small. Find the bound.

(6e) Argue that the Newtonian limit corresponds to the r^{+}_{circular} with the square root replaced by 1.