General Relativity 2011/12 – Final Exam

Fri Feb 3 2012, 9-12am BBL 056+169

This final exam caries in total 20 points.
A passing requirement of the course is that you obtain at least 55% of the points.
In order to qualify for a retake, you need to have at least 40% (or 8 points) at this exam.
Please provide concise answers to the theoretical questions.
You should use a separate sheet of paper for each problem. Put your name on each sheet.
`Good luck!`

Exercise 1 - Theoretical questions (*4 points*)

(A) *(1 point)*

Define the Einstein's equivalence principle!

(B) *(1 point)*

Define the tangent space at a point P of a (pseudo-Riemannian) manifold $\mathcal{M}$! What is the name for the collection of all tangent spaces on a manifold, and what is this construct useful for?

(C) *(1 point)*

Define a singularity in general relativity! How would you distinguish a coordinate singularity from a physical singularity?

(D) *(1 point)*

Can photons move on a periodic orbit around a black hole? If yes, (a) is that orbit circular or elliptic; (b) stable or unstable? Can photons move on a periodic orbit around the Sun?

Exercise 2 - The hyperbolic plane (*4 points*)

The hyperbolic plane is defined by the metric:

$$d\vec{\ell}^{\,2} = \frac{dx^2 + dy^2}{y^2}, \qquad y > 0\,. \tag{1}$$

(A) *(1 point)*

Show that points on the x axis are an infinite distance from any point (x, y) in the upper half-plane! Write out the geodesic equations!

(B) *(1 point)*

Construct a Killing vector field K^μ for the metric (1), and show that it obeys the Killing equation $\nabla_{(\mu}K_{\nu)} = 0$. Next, construct the corresponding conserved quantity, $K_\mu dx^\mu/d\ell$, and give a physical interpretation for it.

(C) *(1 point)*

Solve the geodesic equations to find x and y as a function of the (proper) length ℓ along these curves! Write down the general solution for x and y as a function of ℓ, and explain the meaning of the three integration constants.

(D) *(1 point)*

Show that the geodesics are semi-circles centered on the y-axis (when the conserved momentum $P_x \neq 0$), or vertical lines (when $P_x = 0$)!

Hint: By making use of the appropriate Killing vector, (when solving parts C and D) show first that the momentum in the x-direction, $P_x = (dx/d\ell)/y^2$, is conserved. Next, solve the integrated geodesic equation, $1 = [(x')^2 + (y')^2]/y^2$, where $x' \equiv dx/d\ell$ and $y' \equiv dy/d\ell$, and ℓ denotes a proper length. The following integral you may find useful,

$$\int \frac{du}{\sqrt{u^2-1}} = \text{Arccosh}(u)\,. \qquad (2)$$

Exercise 3 - Conformal transformation of the metric tensor *(4 points)*

Consider the following conformal transformation of the metric,

$$ds^2(x) = e^{2\omega(x)} d\bar{s}^2(x)\,. \qquad (3)$$

(A) *(1 point)*

Show that the scalar d'Alembertian $\Box$ transforms under the conformal transformation (3) as

$$e^{2\omega}\Box\phi(x) = \bar{\Box}\phi(x) + 2(\bar{\nabla}_\mu\omega)(\bar{\nabla}^\mu\phi(x))\,, \qquad (4)$$

where $\phi(x)$ is a scalar field, and $\bar{\nabla}_\mu$ represents the covariant derivative with respect to the rescaled metric $d\bar{s}^2$, and $\bar{\Box} = \bar{\nabla}_\mu\bar{\nabla}^\mu$ is the scalar d'Alembertian associated with the rescaled metric $d\bar{s}^2$.

(B) *(2 points)*

Show that the curvature (Ricci) scalar $\mathcal{R}$ transforms under (3) as

$$e^{2\omega}\mathcal{R} = \bar{\mathcal{R}} - 6(\bar{\nabla}_\mu\omega)(\bar{\nabla}^\mu\omega) - 6\bar{\Box}\omega\,. \qquad (5)$$

(C) *(1 point)*

By making use of the flat cosmological space-time, which is defined by $e^{\omega} \to a(\eta)$ and $d\bar{s}^2 = -d\eta^2 + d\vec{x}^2$, calculate the Ricci scalar for flat cosmological spaces by making use of your results in parts (A) and (B) of this problem. Express your answer in terms of the Hubble parameter $H = a'/a^2$, $a' \equiv da/d\eta$, and $\dot{H} = (a''/a^3) - 2(a'/a^2)^2$.

Hint: Recall the definition of the Riemann curvature tensor in terms of the Levi-Civita connection,

$$\mathcal{R}^{\alpha}{}_{\mu\nu\rho} = \partial_\nu \Gamma^{\alpha}_{\mu\rho} + \Gamma^{\alpha}_{\nu\beta}\Gamma^{\beta}_{\mu\rho} - (\nu \leftrightarrow \rho)\,. \tag{6}$$

When you work on part B, at an intermediate stage, you can work out the problem in geodesic normal coordinates, in which at a given point P, $\bar{g}_{\mu\nu}|_P = \eta_{\mu\nu}$ and $\Gamma^{\alpha}_{\mu\nu}|_P = 0$, where $\eta_{\mu\nu}$ is the flat space (Minkowski) metric. The answer in geodesic normal coordinates is then easily generalised to general coordinates, in which the metric tensor is $\bar{g}_{\mu\nu}$.

Exercise 4 - The conformal (Carter-Penrose) diagram of anti-de Sitter (AdS) space *(4 points)*

Anti-de Sitter space is defined by the following 5 dimensional flat embedding:

$$\begin{aligned} ds_5^2 &= -dU^2 - dV^2 + dX^2 + dY^2 + dZ^2 \\ -\frac{1}{H^2} &= -U^2 - V^2 + X^2 + Y^2 + Z^2\,. \end{aligned} \tag{7}$$

By making use of the following coordinate transformation,

$$\begin{aligned} U &= \frac{1}{H}\sin(Ht)\cosh(H\rho) \\ V &= \frac{1}{H}\cos(Ht)\cosh(H\rho) \\ X &= \frac{1}{H}\sinh(H\rho)\cos(\theta) \\ Y &= \frac{1}{H}\sinh(H\rho)\sin(\theta)\cos(\phi) \\ Z &= \frac{1}{H}\sinh(H\rho)\sin(\theta)\sin(\phi)\,, \end{aligned} \tag{8}$$

(A) *(1 point)*

show that the resulting four dimensional metric on AdS_4 can be written as,

$$ds^2 = -\cosh^2(H\rho)dt^2 + d\rho^2 + \frac{\sinh^2(H\rho)}{H^2}\left(d\theta^2 + \sin^2(\theta)d\phi^2\right)\,. \tag{9}$$

(B) *(1 point)*

Next, by performing the following coordinate transformations,

$$\cosh(H\rho) = \frac{1}{\cos(\chi)}, \qquad Ht = \tau, \tag{10}$$

show that the metric reduces to

$$ds^2 = \frac{1}{H^2\cos^2(\chi)}\left[-d\tau^2 + d\chi^2 + \sin^2(\chi)\left(d\theta^2 + \sin^2(\theta)d\phi^2\right)\right]. \tag{11}$$

(C) *(2 points)*

Based upon the metric (11), discuss the allowed range of the coordinates $\{\tau, \chi, \theta, \phi\}$ and draw the corresponding Carter-Penrose conformal diagram of anti-de Sitter space! Denote the relevant 'infinities' on the diagram (when applicable), such as $i^\pm, i^0$ and $\mathcal{I}^\pm$.

Exercise 5 - A curved universe with cosmological constant *(4 points)*

Consider a universe filled with a vacuum energy (cosmological constant) $\Lambda > 0$ with negatively curved spatial sections, *i.e.* $\kappa < 0$.

(A) *(1 point)*

Solve the corresponding Friedmann equation for the scale factor $a = a(t)$ and find an expression for the Hubble parameter, $H = H(t)$!

(B) *(2 points)*

Find the particle horizon $\ell_{\rm ph} = a\ell_c$ as a function of time, where $\ell_c = \eta - \eta_0$ ($c = 1$) is the maximum distance that a particle can travel from η_0 to $\eta > \eta_0$. Express your answer in cosmological time $t \geq 0$ (defined by $g_{00} = 1$)!

(C) *(1 point)*

In this case, the Friedmann equation can be rewritten simply as $1 = \Omega_\Lambda + \Omega_\kappa$, where $\Omega_\Lambda(t) \equiv \Lambda/(3H^2)$ and $\Omega_\kappa(t) \equiv -\kappa/(aH)^2$. Find an explicit form for $\Omega_\kappa(t)$! How does $\Omega_\kappa(t)$ behave for asymptotically large times, $t \to \infty$?

Hint: When solving the problem, you may find the following integrals useful,

$$\int \frac{dx}{\sqrt{x^2+1}} = \text{Arcsinh}(x), \qquad \int \frac{dx}{\sinh(x)} = \ln\left[\tanh\left(\frac{x}{2}\right)\right]. \tag{12}$$